MAT246H1S Lec0101 Burbulla

Chapter 10 Lecture Notes
Sizes of Infinite Sets

Winter 2017

What Is A Set?

One of the most basic ideas in all of mathematics is that of a **set.** Naively, we just define a set as a collection of objects. We can describe a set in at least two different ways:

1. We can list the members of the set. For example,

$$\mathcal{S} = \{a, b, c\}$$

   is the set that contains the three letters, a, b and c.

2. Or we can state one or more conditions that the members of a set must satisfy. For example, $\mathcal{S}$ could be the set of all prime numbers that are between 1 and 25. Then

$$\mathcal{S} = \{2, 3, 5, 7, 11, 13, 17, 19, 23\},$$

   which can also be written as

$$\mathcal{S} = \{p \mid p \text{ is a prime, and } 1 \leq p \leq 25\}.$$

Subsets and Equality of Sets

If a is in the set $\mathcal{S}$ we write

$$a \in \mathcal{S}.$$

If we have two sets $\mathcal{S}$ and $\mathcal{T}$ and every element in $\mathcal{S}$ is also in $\mathcal{T}$, that is, $a \in \mathcal{S} \Rightarrow a \in \mathcal{T}$, then we call $\mathcal{S}$ a **subset** of $\mathcal{T}$ and we write

$$\mathcal{S} \subset \mathcal{T}.$$

Thus

$$\{0, 3, 6, 9\} \subset \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

and

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Note: every set is a subset of itself. We say two sets are **equal** if they are both subsets of each other; that is

$$\mathcal{S} = \mathcal{T} \Leftrightarrow \mathcal{S} \subset \mathcal{T} \text{ and } \mathcal{T} \subset S.$$

The Empty Set

The set with no members is called the **empty set** and it is denoted by $\emptyset$. The empty set is considered to be a subset of every set. Thus all the subsets of the set $\mathcal{S} = \{a, b, c\}$ are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}.$$

Of course, it may not be obvious at first if a set is the empty set. Consider

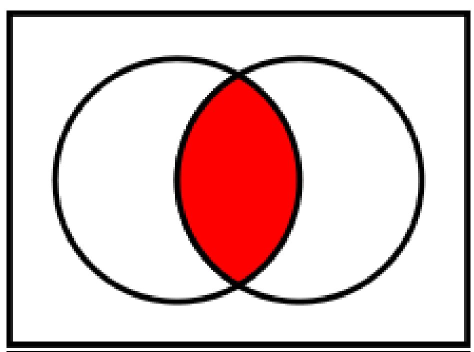
$$\mathcal{S} = \{x \in \mathbb{R} \mid x^2 + x = 0 \text{ and } x > 2\}.$$

But it is empty, since the only solutions to the equation $x^2 + x = 0$ are $x = -1$ and $x = 0$, neither of which is greater than 2. Similarly in linear algebra: not every system of linear equations has a solution; an inconsistent system of equations is simply a system with empty solution set.

The Intersection and Union of Sets

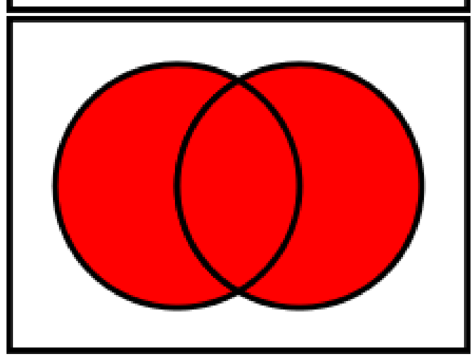
If $\mathcal{S}$ and $\mathcal{T}$ are two sets, we define their **intersection** as the set of elements common to $\mathcal{S}$ and $\mathcal{T}$, and we write it as $\mathcal{S} \cap \mathcal{T}$. That is,

$$\mathcal{S} \cap \mathcal{T} = \{a \mid a \in \mathcal{S} \text{ and } a \in \mathcal{T}\}.$$



If $\mathcal{S}$ and $\mathcal{T}$ are two sets, we define their **union** as the set of elements contained in $\mathcal{S}$ or in $\mathcal{T}$, and we write it as $\mathcal{S} \cup \mathcal{T}$. That is,

$$\mathcal{S} \cup \mathcal{T} = \{a \mid a \in \mathcal{S} \text{ or } a \in \mathcal{T}\}.$$



Example 1

Suppose

$$\mathcal{S} = \{1,2,3,4,5,6,7\}, \mathcal{T} = \{2n \mid n \in \mathbb{Z}\}, \mathcal{U} = \{2n+1 \mid n \in \mathbb{Z}\}.$$

Then

$$\mathcal{S} \cap \mathcal{T} = \{2,4,6\}; \mathcal{S} \cap \mathcal{U} = \{1,3,5,7\}; \mathcal{T} \cup \mathcal{U} = \mathbb{Z}.$$

Note

$$\mathcal{T} \cap \mathcal{U} = \emptyset.$$

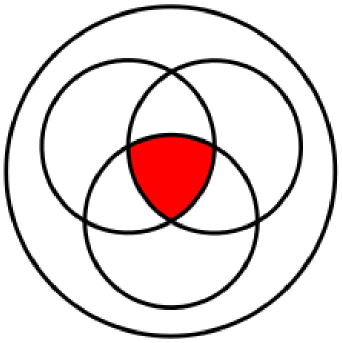
Two sets that have empty intersection are called **disjoint** sets.

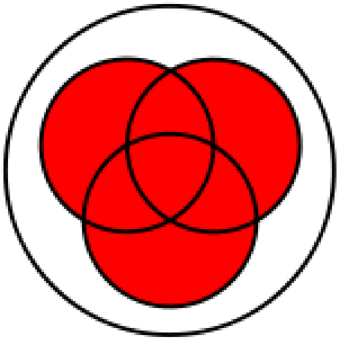
Intersection and Union of More Than Two Sets

You can define the intersection and union or more than two sets:

$$\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3 = \{s \mid s \in \mathcal{S}_1 \text{ and } s \in \mathcal{S}_2 \text{ and } s \in \mathcal{S}_3\},$$

$$\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \{s \mid s \in \mathcal{S}_1 \text{ or } s \in \mathcal{S}_2 \text{ or } s \in \mathcal{S}_3\}.$$





Types of Functions

Suppose $\mathcal{S}$ and $\mathcal{T}$ are sets, and

$$f : \mathcal{S} \longrightarrow \mathcal{T}$$

is a function that maps elements of $\mathcal{S}$ to elements of $\mathcal{T}$. Then

1. $\mathcal{S}$ is called the **domain** of f.
2. $\{f(s) \mid s \in \mathcal{S}\} \subset \mathcal{T}$ is called the **range** of f.
3. f is **one-to-one,** or injective, if $f(s_1) = f(s_2) \Rightarrow s_1 = s_2$.
4. f is **onto,** or surjective, if the range of f is $\mathcal{T}$.
5. f is a **one-to-one correspondence,** or bijective, if it is one-to-one and onto.
6. If f is bijective, the **inverse** of f, denoted by f^{-1}, is defined by

$$f^{-1}(t) = s \Leftrightarrow f(s) = t.$$

Note that $f^{-1} : \mathcal{T} \longrightarrow \mathcal{S}$ and $(f \circ f^{-1})(t) = t$, for all $t \in \mathcal{T}$.

Example 2

Let $\mathcal{S} = (-\pi/2, \pi/2)$, let $\mathcal{T} = \mathbb{R}$. Define $f : \mathcal{S} \longrightarrow \mathcal{T}$ by $f(x) = \tan x$. Then f is one-to-one and onto, and $f^{-1} : \mathcal{T} \longrightarrow \mathcal{S}$ by $f^{-1}(x) = \arctan x$.

Let $\mathcal{S} = \mathcal{T} = \mathbb{R}$; let $g : \mathcal{S} \longrightarrow \mathcal{T}$ be defined by $g(x) = x^2$. Then g is not one-to-one:

$$g(x_1) = g(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2.$$

The range of g is $\{y \in \mathbb{R} \mid y \geq 0\}$.

Cardinality of A Set

Naively, the cardinality of a set is the size of the set, or the number of elements in the set. However, we are going to consider sets with an infinite number of elements, so this approach is too vague. First we define the **cardinality of a finite set:**

A set $\mathcal{S}$ has cardinality n if it can be put into a one-to-one correspondence with the set $\{1, 2, \ldots, n\}$. We write $|\mathcal{S}| = n$, and say $\mathcal{S}$ is **finite.** We define the cardinality of the empty set to be zero, $|\emptyset| = 0$, and consider it to be finite too.

If a non-empty set $\mathcal{S}$ cannot be put into a one-to-one correspondence with a finite set, then it is called an **infinite set.**

Two sets $\mathcal{S}$ and $\mathcal{T}$ have the **same cardinality** if there is a one-to-one correspondence between them, and we write $|\mathcal{S}| = |\mathcal{T}|$.

Example 3

1. If $\mathcal{S} = \{x \in \mathbb{R} \mid x^3 = x\}$, then $\mathcal{S} = \{0, 1, -1\}$ and $|\mathcal{S}| = 3$.
2. If $\mathcal{S} = \{x \in \mathbb{R} \mid x^3 = 1\}$, then $\mathcal{S} = \{1\}$ and $|\mathcal{S}| = 1$.
3. If $\mathcal{S} = \{x \in \mathbb{C} \mid x^3 = 1\}$, then $\mathcal{S} = \{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\}$ and $|\mathcal{S}| = 3$.
4. Let A be an $m \times n$ matrix, let $\vec{b} \in \mathbb{R}^m$, let

$$\mathcal{S} = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{b}\}.$$

Then $|\mathcal{S}| = 0$, if the system of linear equations is inconsistent; $|\mathcal{S}| = 1$ if the system has a unique solution; otherwise $\mathcal{S}$ is an infinite set.

Example 4

Let

$$\mathcal{E} = \{2, 4, 6, \dots\} = \{2n \mid n \in \mathbb{N}\}, \mathcal{O} = \{1, 3, 5, \dots\} = \{2n-1 \mid n \in \mathbb{N}\}.$$

Then both $\mathcal{E}$ and $\mathcal{O}$ are infinite sets, and $|\mathcal{E}| = |\mathcal{O}|$. To see this, define

$$f : \mathcal{O} \longrightarrow \mathcal{E}$$

by $f(n) = n + 1$. Then

- f is one-to-one: $f(n_1) = f(n_2) \Rightarrow n_1 + 1 = n_2 + 1 \Rightarrow n_1 = n_2$
- f is onto: if $2k \in \mathcal{E}$, take $n = 2k - 1$. Then $n \in \mathcal{O}$ and
$$f(n) = f(2k - 1) = 2k - 1 + 1 = 2k.$$

So f is a one-to-one correspondence between $\mathcal{O}$ and $\mathcal{E}$, and hence $|\mathcal{O}| = |\mathcal{E}|$. There are just as many odd numbers as even numbers.

Example 5

More surprisingly (perhaps): there are just as many even numbers as natural numbers, even though $\mathcal{E} \subset \mathbb{N}$ and $\mathcal{E} \neq \mathbb{N}$. To see this, define $f : \mathbb{N} \longrightarrow \mathcal{E}$ by $f(n) = 2n$. It is easy to show that f is one-to-one and onto. In the same vein, there are just as many perfect squares as natural numbers. Let

$$\mathcal{S} = \{1, 4, 9, 16, \dots\} = \{n^2 \mid n \in \mathbb{N}\}.$$

Define $g : \mathbb{N} \longrightarrow \mathcal{S}$ by

$$g(n) = n^2.$$

This is clearly onto; is it one-to-one? Yes: $n_1^2 = n_2^2 \Rightarrow n_1 = \pm n_2$, but n_1 and n_2 are both positive, so $n_1 = n_2$. Thus g is one-to-one and onto, and $|\mathcal{S}| = |\mathbb{N}|$.

The Cardinality of the Rational Numbers

Theorem 10.1.14: the set of natural numbers, $\mathbb{N}$, and the set of positive rational numbers, $\mathbb{Q}^+$, have the same cardinality.

Proof: the proof involves listing the positive rational numbers in an organized way:

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{6}{6} & \cdots
\end{array}$$

Define

$$f : \mathbb{N} \longrightarrow \mathbb{Q}^+$$

by 'counting' the rationals, starting from the top left. $f(1) = 1, f(2) = 1/2,$ $f(3) = 2, f(4) = 3, f(5) = ?$

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10.1: Cardinality
10.2: Countable Sets and Uncountable Sets
10.3: Comparing Cardinalities

The next rational number in our 'zig-zag' path is 2/2, but this is the same as 1, which we have already counted, so we take $f(5) = 1/3$, and then $f(6) = 1/4$. Then going down the diagonal from top right to bottom left we get

$$f(7) = \frac{2}{3}, f(8) = \frac{3}{2}, f(9) = 4, f(10) = 5.$$

Now go up the diagonal from 5: $4/2 = 2$ has already been counted, $3/3 = 1$ has already been counted, $2/4 = 1/2$ has already been counted, so take $f(11) = 1/5$, and then take $f(12) = 1/6$. In this way we can establish a one-to-one correspondence between the natural numbers and the positive rational numbers. Thus

$$|\mathbb{N}| = |\mathbb{Q}^+|.$$

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What Is A Countable Set?

Definition: a set is *countable* if it is finite or if it has the same cardinality as $\mathbb{N}$. A set is called *uncountable* if it is not countable.

The infinite sets we looked at in Section 10.1 all had the same cardinality as $\mathbb{N}$, so they are all countable. **Question:** are there sets that are uncountable? The answer is "yes," as Georg Cantor showed in 1891. In particular, Cantor showed that the set of real numbers between 0 and 1 is uncountable. To do this he used what has become known as a 'diagonalization' argument. Let

$$\mathcal{S} = \{s \in \mathbb{R} \mid 0 \leq s \leq 1\} = [0, 1].$$

Observe that if $s \in \mathcal{S}$, then it can be written as a never-ending decimal,

$$s = 0.a_1 a_2 a_3 \ldots a_j \ldots,$$

with each $a_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What does this mean?

Base 10 Decimal Expansions

If $s = 0.a_1 a_2 a_3 \ldots a_j \ldots$, this means

$$s = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_j}{10^j} + \cdots = \sum_{j=1}^{\infty} \frac{a_j}{10^j}.$$

Some numbers have two decimal expansions. For example, $0.1 = 0.09999999\ldots$ since

$$\sum_{j=2}^{\infty} \frac{9}{10^j} = \frac{9}{100} \sum_{j=0}^{\infty} \frac{1}{10^j} = \frac{9}{100} \frac{1}{1 - 1/10} = \frac{9}{100} \cdot \frac{10}{9} = \frac{1}{10}.$$

Thus, we can always rewrite a finite decimal expansion as an infinite decimal expansion, by replacing an infinite string of 0's with an infinite string of 9's.

Binary 'Decimal' Expansions

Of course, there is nothing special about base 10. Any base could be used; indeed Cantor's proof is simpler if we switch to base 2, or binary, 'decimal' expansions. That is, if $s \in [0, 1]$, we can write it as $s = 0.b_1 b_2 \dots b_j \dots$ with each $b_j \in \{0, 1\}$ and

$$s = \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_j}{2^j} + \cdots = \sum_{j=1}^{\infty} \frac{b_j}{2^j}.$$

As with base 10, some numbers have two different binary decimal expansions. For example,

$$\frac{1}{2} = 0.1 \text{ or } \frac{1}{2} = 0.01111111\dots,$$

as you can check. So every $s \in [0, 1]$ can be written as an infinite binary decimal expansion.

Theorem 10.2.2: The Closed Interval $[0, 1]$ is Uncountable

Proof: Suppose $[0, 1]$ is countable; then there is a bijection $f : \mathbb{N} \longrightarrow [0, 1]$. List the elements of $[0, 1]$ as $f(1), f(2), f(3), \dots$:

$$\begin{array}{lll} f(1) & = & 0.\mathbf{b_{11}} b_{12} b_{13} b_{14} b_{15} \dots \text{ with } b_{1i} \in \{0,1\} \\ f(2) & = & 0.b_{21} \mathbf{b_{22}} b_{23} b_{24} b_{25} \dots \text{ with } b_{2i} \in \{0,1\} \\ f(3) & = & 0.b_{31} b_{32} \mathbf{b_{33}} b_{34} b_{35} \dots \text{ with } b_{3i} \in \{0,1\} \\ f(4) & = & 0.b_{41} b_{42} b_{43} \mathbf{b_{44}} b_{45} \dots \text{ with } b_{4i} \in \{0,1\} \\ f(5) & = & 0.b_{51} b_{52} b_{53} b_{54} \mathbf{b_{55}} \dots \text{ with } b_{5i} \in \{0,1\} \\ \dots & \dots & \qquad\qquad\qquad \dots \end{array}$$

Define $b = 0.b_1 b_2 b_3 b_4 b_5 \dots$ such that $b_i = \begin{cases} 0, & \text{if } b_{ii} = 1 \\ 1, & \text{if } b_{ii} = 0 \end{cases}$.

Then $b \in [0, 1]$, but b is not on the above list because it differs with each listed binary expansion in at least one digit. That is, there is a $b \in [0, 1]$ that is not in the range of f. This contradicts the assumption that f is onto. Thus $[0, 1]$ is uncountable.

The Closed Interval $[a, b]$ Is Also Uncountable

Theorem 10.2.4: if $a < b$ then $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is uncountable.

Proof: we need only show that $[a, b]$ and $[0, 1]$ have the same cardinality. To do this we need to find a bijection between $[0, 1]$ and $[a, b]$. One such correspondence is $f : [0, 1] \longrightarrow [a, b]$ defined by

$$f(x) = a + (b - a)x.$$

We need to check f is one-to-one and onto. We can use calculus! f is one-to-one because $f'(x) = b - a > 0$, which means f is an increasing, hence one-to-one, function. f is continuous and $f(0) = a$ and $f(1) = b$. If $y \in (a, b)$, there is a $c \in (0, 1)$ such that $f(c) = y$, by the Intermediate Value Theorem. Thus f is onto.

$(0, 1]$ and $[0, 1]$ Have The Same Cardinality

Theorem 10.2.6: $|[0, 1]| = |(0, 1]|$.

Proof: we have to construct a one-to-one function $f : [0, 1] \longrightarrow (0, 1]$ that is also onto. We can patch together a function as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 1/(n+1), & \text{if } x = 1/n, \text{ for } n \in \mathbb{N} \\ x, & \text{otherwise} \end{cases}$$

First note that 0 is not in the range of f, since $1/k \neq 0$, for $k \in \mathbb{N}$, and $f(x) = x$ doesn't apply if $x = 0$. Of course, you still have to check that f is one-to-one, and onto, which are straightforward enough.

Transitivity of Cardinality

Theorem 10.2.7: if $|\mathcal{S}| = |\mathcal{T}|$ and $|\mathcal{T}| = |\mathcal{U}|$, then $|\mathcal{S}| = |\mathcal{U}|$.

Proof: since $|\mathcal{S}| = |\mathcal{T}|$ there is a bijection $f : \mathcal{S} \longrightarrow \mathcal{T}$; since $|\mathcal{T}| = |\mathcal{U}|$ there is a bijection $g : \mathcal{T} \longrightarrow \mathcal{U}$. We claim

$$g \circ f : \mathcal{S} \longrightarrow \mathcal{U}$$

is also a bijection:

$g \circ f$ is one-to-one:
$g(f(s_1)) = g(f(s_2)) \Rightarrow f(s_1) = f(s_2)$ (Why?) $\Rightarrow s_1 = s_2$ (Why?)

$g \circ f$ is onto:

let $u \in \mathcal{U}$. Since g is onto, there is a $t \in \mathcal{T}$ such that $g(t) = u$.
Since f is onto, there is $s \in \mathcal{S}$ such that $f(s) = t$.
Then $g(f(s)) = u$.

$(a, b]$ and $(c, d]$ Have The Same Cardinality

Theorem 10.2.8: if $a < b$ and $c < d$, then $|(a, b]| = |(c, d]|$.

Proof: we use Theorem 10.2.7.

$f : (0, 1] \longrightarrow (a, b]$ defined by $f(x) = a + (b - a)x$ is a bijection, so

$$|(0, 1]| = |(a, b]|.$$

$g : (0, 1] \longrightarrow (c, d]$ defined by $g(x) = c + (d - c)x$ is a bijection, so

$$|(0, 1]| = |(c, d]|.$$

Then by transitivity of cardinality,

$$|(a, b]| = |(c, d]|.$$

The Nonnegative Real Numbers Are Uncountable

Theorem 10.2.9: the cardinality of the set of nonnegative real numbers is the same as the cardinality of the interval $[0, 1]$.

Proof: the solution in the book is one way; here is another way. Define

$$f : [0, 1) \longrightarrow \{y \in \mathbb{R} \mid y \geq 0\}$$

by

$$f(x) = \tan\left(\frac{\pi x}{2}\right).$$

From your calculus course you should know that f is one-to-one and onto. So $|[0, 1)| = |\{y \in \mathbb{R} \mid y \geq 0\}|$. Finally, in a fashion very similar to Theorem 10.2.6, we can show that $|[0, 1)| = |[0, 1]|$.

A Countable Union of Countable Sets is Countable

Theorem 10.2.10: if $\mathcal{S}_n$, for $n \in \mathbb{N}$, is a countable set, then

$$\bigcup_{n=1}^{\infty} S_n$$

is also countable.

Proof: outline. We use a zig-zag argument, as in the proof of Theorem 10.1.14. List the elements in each set,

$$\begin{array}{lcl} S_1 & = & \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \ldots\} \\ S_2 & = & \{a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \ldots\} \\ S_3 & = & \{a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \ldots\} \\ \ldots & & \ldots \end{array}$$

Now start counting them all from the top left, skipping repetitions.