Integration

Recall, from MAT A32, the notion of anti-differentiation.

Defn: $F(x)$ is an ANTIDERIVATIVE of $f(x)$ if
$$F'(x) = f(x).$$

Ex: $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.
$F(x) = \frac{1}{3}x^3 + 10$ is also an antiderivative of x^2.

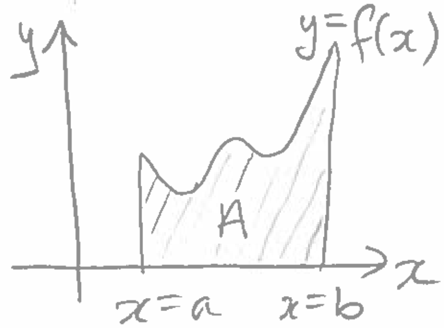
Fact: If $F(x)$ and $g(x)$ are two anti-derivatives of $f(x)$ then $F(x) - g(x) = C$ is constant.

The Fundamental Theorem of Calculus:

If $f(x) \geq 0$ between $x=a$ and $x=b$ and $F(x)$ is an antiderivative of $f(x)$

then

the area under the curve $y = f(x)$ between $x=a$ and $x=b$ is $F(b) - F(a)$



$A = F(b) - F(a)$

Integration

Nota: we write $F(x)+C = \int f(x)\,dx$ for
" $F(x)+C$ is the anti-derivative of $f(x)$."
This is called an INDEFINITE INTEGRAL.

We write $\int_a^b f(x)\,dx = F(b)-F(a)$ for the SIGNED AREA of $y=f(x)$ from $x=a$ to $x=b$.
This is called a DEFINITE INTEGRAL.

Ex: Find the area under the curve $y=x^2$ from $x=1$ to $x=2$.

$$\int_1^2 x^2 = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}\cdot 2^3 - \frac{1}{3}\cdot 1^3 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

Thm: If $k \neq -1$ then $\int x^k\,dx = \frac{1}{k+1}x^{k+1} + C$.

Thm (Linearity) $\int af(x)+bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx$

Ex: Calculate $\int_0^1 \sqrt{x} + 5x^4 + x^{\frac{1}{3}}\,dx$

$$= \left[\frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + 5\cdot\frac{1}{4+1}x^{4+1} + \frac{1}{\frac{1}{3}+1}x^{\frac{1}{3}+1}\right]_0^1$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + x^5 + \frac{3}{4}x^{\frac{4}{3}}\right]_0^1$$

$$= \frac{2}{3} + 1 + \frac{3}{4} = \frac{8}{12} + \frac{12}{12} + \frac{9}{12} = \frac{29}{12}.$$

Substitution

Ex: Calculate $\int [\sqrt{2x+1}]\cdot 2\,dx$

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$du = 2dx$

$$= \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}(2x+1)^{\frac{3}{2}} + C.$$

Ex: Calculate $\int (\sin(x)+2)^3 \cos(x)\,dx$

Let $u = \sin(x)+2$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)dx$

$$= \int u^3 du$$

$$= \frac{1}{4}u^4 + C$$

$$= \frac{1}{4}(\sin(x)+2)^4 + C.$$

Ex: Calculate $\int x e^{x^2}\,dx$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$$= \frac{1}{2}\int 2x e^{x^2}\,dx$$

$$= \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2} + C.$$

Iterated Integrals

Observation: If $f(x,y)$ is a function of two variables then $\int_a^b f(x,y)\,dx$ is a function of one variable.

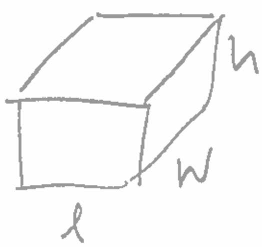
$$g(y) = \int_a^b f(x,y)\,dx$$

Thus we can integrate:

$$\text{volume} = \int_c^d g(y)\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Ex: Find the volume of a $l \times w \times h$ cube

$$V = \int_0^w \int_0^l h\,dx\,dy = \int_0^w [hx]_0^l\,dy$$



$$= \int_0^w hl\,dy$$

$$= [hly]_0^w = hlw.$$

$$= lwh.$$

Ex: Find the volume under $z = x^2 + y^2$ above the patch $x=1$ $x=2$ and $y=0$ $y=1$.

$$V = \int_0^1 \int_1^2 x^2 + y^2\,dx\,dy = \int_0^1 \left[\tfrac{1}{3}x^3 + y^2 x\right]_1^2 dy$$

$$= \int_0^1 \tfrac{7}{3} + y^2\,dy = \left[\tfrac{7}{3}y + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{7}{3} + \tfrac{1}{3} = \tfrac{8}{3}.$$

<u>Ex</u> (Winter 2014) : Evaluate $\int_0^4 \int_0^2 (x+\sqrt{2y+1})\,dx\,dy$

$$= \int_0^4 \left[\tfrac{1}{2}x^2 + x\sqrt{2y+1}\right]_0^2 dy$$

$$= \int_0^4 2 + 2\sqrt{2y+1}\,dy$$

$$\int 2+2\sqrt{2y+1}\,dy$$
$$= \int 2\,dy + \int 2\sqrt{2y+1}\,dy$$
$$= 2y + \tfrac{2}{3}(2y+1)^{\frac{3}{2}} + C$$

$$= \left[2y+\tfrac{2}{3}(2y+1)^{\frac{3}{2}}\right]_0^4$$

$$= 2\cdot 4 + \tfrac{2}{3}\cdot(2\cdot 4+1)^{\frac{3}{2}}$$

$$= 8 + \tfrac{2}{3}\cdot 9^{\frac{3}{2}} = 8 + \tfrac{2}{3}\cdot 27$$

$$= 8 + 2\cdot 9 = 8+18 = 26.$$

<u>Ex</u> (Winter 2013): Evaluate $\int_0^4 \int_0^1 (3\sqrt{x} + 2y)\,dy\,dx$

$$= \int_0^4 \left[3y\sqrt{x} + y^2\right]_0^1 dx$$

$$= \int_0^4 (3\sqrt{x}+1)\,dx$$

$$= \left[3\cdot\tfrac{2}{3}x^{\frac{3}{2}} + x\right]_0^4$$

$$= \left[2\cdot x^{\frac{3}{2}} + x\right]_0^4$$

$$= 16 + 4 = 20$$