MAT A33 Wk 9c

Ex: Find all points where $f(x,y) = x^2 - 3x + 2 + y^2 + 9x - 10$ has a horizontal tangent plane

\# calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = 2x - 3 \qquad \frac{\partial f}{\partial y} = 2y + 9$$

\# solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$2x - 3 = 0 \qquad 2y + 9 = 0$$
$$\Rightarrow x = \frac{3}{2} \qquad \Rightarrow y = -\frac{9}{2}$$

Thus, $f(x,y)$ has a horizontal tangent plane at $(x,y) = \left(\frac{3}{2}, -\frac{9}{2}\right)$

Ex: Find all points where $f(x,y) = x^2 - y^2 - xy$ has a horizontal tangent plane

\# calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = 2x - y \qquad \frac{\partial f}{\partial y} = -2y - x$$

\# Solve $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$$\begin{cases} 2x - y = 0 \\ -2y - x = 0 \end{cases} \Rightarrow \begin{cases} 2x - y = 0 \\ x + 2y = 0 \end{cases} \Rightarrow (x,y) = 0$$

Thus, the only horizontal tangent plane is $(0,0)$.

Ex (p765): If $z = \ln(x^2+y^2)$ show

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

(!) symmetry.

check $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{(-2x^2+2y^2)+(2x^2-2y^2)}{(x^2+y^2)^2}$$

$$= 0.$$

Ex: If $z^2 - 3x^2 + y^2 = 0$ find $\frac{\partial^2 z}{\partial y^2}$.

$$\frac{\partial}{\partial y}(z^2 - 3x^2 + y^2) = \frac{\partial}{\partial y}(0)$$

$$\Rightarrow 2z\frac{\partial z}{\partial y} - 0 + 2y = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\Rightarrow \frac{\partial}{\partial y}\left[2z\frac{\partial z}{\partial y} + 2y\right] = \frac{\partial}{\partial y}[0]$$

$$\Rightarrow 2\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial y} + 2z\frac{\partial^2 z}{\partial y} + 2 = 0$$

Ex: If $P = x^2 + y^2 + z^2$ and $x = a + 2b$
$y = 3a + 4b$
$z = 5a + 6b$

find $\frac{\partial P}{\partial a}$ at $(a, b) = (1, 1)$

Apply the chain rule

$$\frac{\partial P}{\partial a} = \frac{\partial P}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial P}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial a}$$

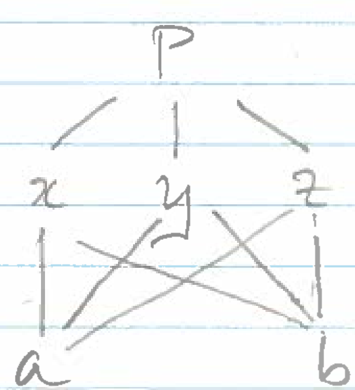
$$= (2x)(1) + (2y)(3) + (2z)(5)$$

Calculate (x, y, z)

$x = 1 + 2 = 3 \quad y = 3 + 4 = 7 \quad z = 5 + 6 = 11$

Evaluate $\frac{\partial P}{\partial a}$

$$\frac{\partial P}{\partial a} = 2 \cdot 3 \cdot 1 + 2 \cdot 7 \cdot 3 + 2 \cdot 11 \cdot 5$$
$$= 6 + 42 + 110 = 158$$



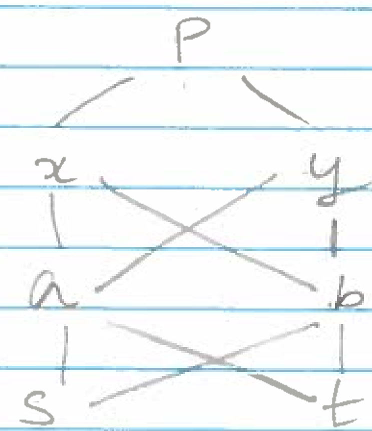
To find $\frac{\partial P}{\partial a}$ we find all paths from a to P.

Ex: If $P(x,y,z) = xy + y^2$

$$x = 2a + 3b \qquad a = s + t$$
$$y = a^2 \qquad b = s$$

Find $\dfrac{\partial P}{\partial s}$ in terms of s and t

Draw the dependence of variables diagram.



$$\frac{\partial P}{\partial s} = \frac{\partial P}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial P}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{\partial P}{\partial x}\left(\frac{\partial x}{\partial a}\frac{\partial a}{\partial s} + \frac{\partial x}{\partial b}\cdot\frac{\partial b}{\partial s}\right) + \frac{\partial P}{\partial y}\left(\frac{\partial y}{\partial a}\cdot\frac{\partial a}{\partial s} + \frac{\partial y}{\partial b}\cdot\frac{\partial b}{\partial s}\right)$$

$$= [y](2\cdot 1 + 3\cdot 1) + [x+2y](2a\cdot 1 + 0\cdot 1)$$

$$= 5y + [x+2y]\cdot(2a)$$
$$= 5(s+t)^2 + [s+t+2(s+t)^2]\cdot(2(s+t))$$

Ex: Assume $P = x + y^2 + z^3$

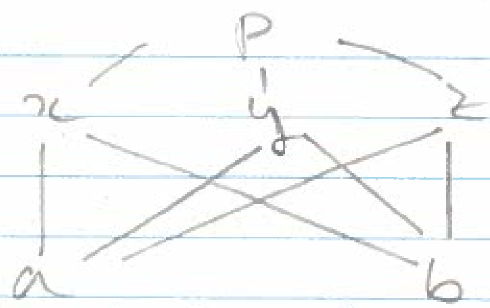
$x = a + 2b$

$y = 2a + 3b$

$z = 3a + 4b$

Calculate $\frac{\partial P}{\partial a}$ at $(a,b) = (1,1)$

Draw the dependence of variables diagram



Apply the chain rule.

$$\frac{\partial P}{\partial a} = \frac{\partial P}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial P}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial a}$$

$$= 1 \cdot (1) + 2y \cdot (2) + 3z^2 (3)$$

$$= 1 + 4y + 9z^2$$

Calculate (x, y, z)

$x = 1 + 2 = 3 \qquad y = 2 + 3 = 5 \qquad z = 3 + 4 = 7$

Evaluate $\frac{\partial P}{\partial a}$

$$\frac{\partial P}{\partial a} = 1 + 4 \cdot 5 + 9 \cdot 7^2 = 1 + 20 + 9 \cdot 49 = 462$$

MAT A33 Wk 9b

The Chain Rule

Recall, from A32, the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$
$$= \frac{df}{dg} \cdot \frac{dg}{dx}$$

Ex: Calculate $\frac{dy}{dx}$ if $y = \sin(t)$ and $t = x^2$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \cos(t) \cdot (2x)$$
$$= \cos(x^2) \cdot 2x.$$

Alternatively, $\frac{dy}{dx} = \frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot (2x)$

Ex: Calculate $\frac{dh}{dx}$ if $h = f^2$, $f = 3g$, $g = \cos(x)$

$$\frac{dh}{dx} = \frac{dh}{df} \cdot \frac{df}{dg} \cdot \frac{dg}{dx}$$
$$= (2f)(3)(-\sin(x))$$
$$= [6\cos(x)]\, 3 \cdot (-\sin(x)) = -18\sin(x)\cos(x)$$

Alternatively, $h = [3\cos(x)]^2$

$$\frac{dh}{dx} = 2 \cdot [3\cos(x)](-3\sin(x))$$
$$= -18\sin(x)\cos(x)$$

Ex: You produce bicycles in china ad India. To make bicycles you need rubber and aluminum.

Describe how an increase in the global availability of aluminum effects your productivity.

$$\begin{tikzcd}
 & P & \\
\text{India} \arrow[ur] & & \text{China} \arrow[ul] \\
\text{Aluminum} \arrow[u] \arrow[urr] & & \text{Rubber} \arrow[u] \arrow[ull]
\end{tikzcd}$$

More aluminum $\Rightarrow$ more in China ad more in India

$\Rightarrow$ more bicycles.

Ex: Suppose $P(x,y) = x + y$ and
$x = 3a + 4b$ and
$y = 5a + 6b$

Determine $\frac{\partial P}{\partial a}$.

$$P = [3a + 4b] + [5a + 6b]$$

$$\frac{\partial P}{\partial a} = 3 + 5 = 8.$$

Ex: Suppose $P(x,y) = x^2 + y^3$

$x = 3a + 4b$

$y = 5a + 6b$

Determine $\frac{\partial P}{\partial a}$.

$$P = (3a+4b)^2 + (5a+6b)^3$$

$\pm$ Differentiate both sides with respect to a

$$\frac{\partial P}{\partial a} = \frac{\partial}{\partial a}(3a+4b)^2 + \frac{\partial}{\partial a}(5a+6b)^3$$

$$= 2(3a+4b)\cdot(3) + 3(5a+6b)^2\cdot(5)$$

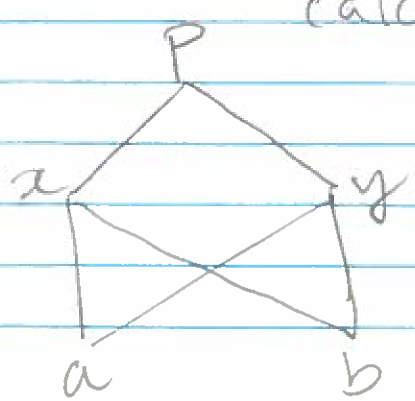
$$= \frac{\partial P}{\partial x}\cdot\frac{\partial x}{\partial a} + \frac{\partial P}{\partial y}\cdot\frac{\partial y}{\partial a}$$

Rule: To calculate $\frac{\partial P}{\partial a}$ given

$P = P(x,y)$

$x = f(a,b) \quad y = g(x,b)$

calculate



$$\frac{\partial P}{\partial a} = \frac{\partial P}{\partial x}\cdot\frac{\partial x}{\partial a} + \frac{\partial P}{\partial y}\cdot\frac{\partial y}{\partial a}$$

Ex (Summer 2016) Given $z = 3x^2 + 2xy$

$x = r^2 + s$

$y = \frac{s}{r+1}$

Calculate $\frac{\partial z}{\partial r}$ when $r=1$ and $s=1$

apply the chain rule

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= [6x+2y](2r) + [2x]\left(-\frac{s}{(r+1)^2}\right)$$

calculate x and y

$x = r^2 + s = 1^2 + 1 = 2$

$y = \frac{1}{1+1} = \frac{1}{2}$

Evaluate $\frac{\partial z}{\partial r}$

$$\frac{\partial z}{\partial r} = [6\cdot 2 + 2\cdot\tfrac{1}{2}](2\cdot 1) + [2\cdot 2]\cdot\left(-\frac{1}{(1+1)^2}\right)$$

$$= [13](2) + [4]\left(-\tfrac{1}{4}\right) = 25$$

NB: We need to use r and s to calculate x and y.

$$(r,s) \longrightarrow (x,y) \longrightarrow \frac{\partial z}{\partial r}$$

Ex (Winter 2015) given $z = (3x-4y)^3$

$x = r + s^2$

$y = 2r^2 + s$

calculate $\frac{\partial z}{\partial r}$ when $r=1$ and $s=2$.

apply the chain rule

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= [3(3x-4y)^2 \cdot (3)] \cdot (1) +$$

$$[3(3x-4y)^2 \cdot (-4)] (4r)$$

calculate x and y

$x = 1 + 2^2 = 5 \qquad y = 2 \cdot 1^2 + 2 = 4$

evaluate $\frac{\partial z}{\partial r}$

$$\frac{\partial z}{\partial r} = [3(3 \cdot 5 - 4 \cdot 4)^2 \cdot (3)](1) +$$

$$[3 \cdot (3 \cdot 5 - 4 \cdot 4)^2 \cdot (-4)] (4 \cdot 1)$$

$$= [3 \cdot (-1)^2 \cdot (3)](1) + [3(-1)^2 \cdot (-4)](4)$$

$$= 9 - 48 = -37$$

Ex (Summer 2014):

Given $u = f(r^2 - s^2, s^2 - r^2)$

Show: $s\frac{\partial u}{\partial r} + r\frac{\partial u}{\partial s} = 0$

introduce variables for f

$u = f(x, y)$ where $x = r^2 - s^2$ and $y = s^2 - r^2$

calculate $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cdot(2r) + \frac{\partial u}{\partial y}\cdot(-2r)$$

! We do not know $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial y}$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{\partial u}{\partial x}\cdot(-2s) + \frac{\partial u}{\partial y}\cdot(2s)$$!

calculate $s\frac{\partial u}{\partial r} + r\frac{\partial u}{\partial s}$

$$s\frac{\partial u}{\partial r} + r\frac{\partial u}{\partial s} = s\left[\frac{\partial u}{\partial x}(2r) + \frac{\partial u}{\partial y}(-2r)\right]$$

$$+ r\left[\frac{\partial u}{\partial x}(-2s) + \frac{\partial u}{\partial y}(2s)\right]$$

$$= \frac{\partial u}{\partial x}(2rs - 2rs) + \frac{\partial u}{\partial y}(-2rs + 2rs)$$

$$= 0$$