Parker Glynn-Adey Wednesday March 27th 2024

What Does The $\sin(x)$ Button Do?

Name: ____________________

Goals for This Workshop

- Explain how the $\sin(x)$ button on a calculator calculates sine of x.
- Create "designer" polynomials.
- Learn about univeristy mathematics.

What Does The Calculator Do?

Q1. Get out a calculator and calculate $\sin(1.2)$ where 1.2 is measured in radians.
(Note: If you get $\sin(1.2) \approx 0.0209$ then your calculator is in degrees mode.)

$$\sin(1.2) \approx \underline{\hspace{4cm}}$$

Q2. What do you think your calculator does when it calculates $\sin(x)$? How does it do it?

Designer Polynomials

Q3. Consider the polynomial $p(x) = 2 + \frac{2}{1 \cdot 2}x^2 + \frac{4}{1 \cdot 2 \cdot 3}x^3$.
These fractions are left "uncancelled" and "unmultiplied" for a good reason: to help spot patterns.
Evaluate the following:

(a) $p(0) =$ ____________________

(b) $p'(0) =$ ____________________

(c) $p''(0) =$ ____________________

(d) $p^{(3)}(0) =$ ____________________ $\longleftarrow$ This $p^{(3)}(x)$ is the third derivative of $p(x)$.

Q4. If you wanted a polynomial $p(x)$ so that $p^{(n)}(0) = C$ and $p^{(k)}(0) = 0$ for $k \neq n$.
To put it another way: the n'th derivative of $p(x)$ is C and all other derivatives are zero.
How would you build it? What is the formula for $p(x)$?

$$p(x) = \underline{\hspace{4cm}}$$

Suggestion: If this question is too abstract, pick your favourite numbers n and C.

The Sine Function

Q5. Consider the function $f(x) = \sin(x)$. Calculate the first few derivatives $f^{(n)}(0)$.

(a) $f'(x) =$ ________________ $f'(0) =$ ________________

(b) $f''(x) =$ ________________ $f''(0) =$ ________________

(c) $f^{(3)}(x) =$ ________________ $f^{(3)}(0) =$ ________________

(d) $f^{(4)}(x) =$ ________________ $f^{(4)}(0) =$ ________________

Q6. Do you notice any pattern in the values of the n'th derivative of $\sin(x)$ at $x = 0$?

Let $f(x) = \sin(x)$. If $n = 2k + 1$ is odd then $f^{(n)}(0) =$ ________________.

> And now we have all the pieces that we need for the Big Idea. Suppose that two functions $f(x)$ and $p(x)$ have same value at $x = 0$. That is: $f(0) = p(0)$. Furthermore, suppose that their first derivatives agree at $x = 0$. As a formula: $f'(0) = p'(0)$. Let's go really wild and suppose that $f^{(n)}(0) = p^{(n)}(0)$ for all n. These functions will be *very* similar: $p(x)$ will approximate $f(x)$.

Q7. Let $f(x) = \sin(x)$ as above. Use Q4 and Q6 to design a polynomial $p(x)$ of degree seven so that:

$$f^{(n)}(0) = p^{(n)}(0) \text{ for all } n \leq 7.$$

Put your answer here:

$p(x) =$ _____ $+$ _____ $x+$ _____ x^2+ _____ x^3+ _____ x^4+ _____ x^5+ _____ x^5+ _____ x^6+ _____ x^7

Q8. Use your polynomial from Q7 to estimate $\sin(1.2)$ and compare it with your answer from Q1.

$p(1.2) \approx$ ____________________ $\sin(1.2) \approx$ ____________________

Q9. What *is* the calculator doing when you hit $\sin(x)$?

Q10. **Further Exploration**

(a) Use Desmos to plot a polynomial of degree N that approximates $\sin(x)$. You will want to type `sum` to get a summation sign $\sum_{n=0}$ and $n!$ for the factorial $n! = 1 \cdot 2 \cdot 3 \cdots (n-1)n$.

(b) Can you do this polynomial approximation method for other functions? How about $\cos(x)$? Or e^x?

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