Problem Solving Group

Discrete Mathematics [Combinatorics]

Malhar Pandya

June 13, 2023

Contents

1 Warming Up

Nostalgia All questions in this section can be answered based on MATA67.

Question 1.1 (Putnam A2 2020) *Let k be a nonnegative integer. Evaluate*

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}$$

Question 1.2 (Putnam A1 2013) *Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.*

Fun Fact! Pigeons are intelligent, probably understand personal space.

Question 1.3 (Putnam A4 2006) *Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a* <u>*local maximum*</u> *at $k \in S$ if*

(i) $\pi(k) > \pi(k+1)$ *for* $k = 1$*;*

(ii) $\pi(k-1) < \pi(k)$ *and* $\pi(k) > \pi(k+1)$ *for* $1 < k < n$*;*

(iii) $\pi(k-1) < \pi(k)$ *for* $k = n$*.*

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Question 1.4 (Putnam A3 1996) *Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.*

Symmetry! Sometimes friend, sometimes foe.

2 Diving Deeper

Simplify! Some problems bark, but don't bite.

Question 2.1 (Putnam B6 2018) *Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most*

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

Question 2.2 (Putnam B4 2005) *For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.*

Question 2.3 (IMO P1 1972) *Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.*

Be Aware! Some problems just have an air of combinatorics

Question 2.4 (Putnam B3 2012) *A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?*