Problem Solving Group

Playing with problems together.

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Contents:

1 General Problem Solving Advice

Make it easier Often, a contest problem is "hidden" or "veiled" by a layer of additional complexity that is easily removed. Reducing to a simpler problem can remove that layer.

Question 1.1 (Putnam B1 1990) *Find all real-valued continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that:*

$$(f(x))^2 = \int_0^x \left[(f(t))^2 + (f'(t))^2\right]\ dt + 1990$$

Get your hands dirty One thing that often helps with a discrete question is to compute some cases. If you are lucky, surprising patterns emerge that you can exploit to solve the problem.

Question 1.2 (Zeitz 2.2.11) *Consider the function $f : \mathbb{N} \to \mathbb{N}$ such that: $f(1) = 1, f(2n) = f(n), f(2n+1) = f(2n) + 1$. Give an algorithm (at most one sentence long) for computing this function.*

2 Division

In this mini-section, I give an example of a problem that I wanted to solve. The solution involved a significant amount of messing around with cases. The three questions follow my path of "looking for a simpler problem".

Question 2.1 (Bollobás – Cambridge p.49) *Show that every rational number $0 < r < 1$ can be written as a sum of reciprocals of distinct natural numbers. That is, for any $0 < r < 1$ there are $n_1 < n_2 < \cdots < n_N \in \mathbb{N}$ such that:*

$$r = \sum_{k=1}^{N} \frac{1}{n_k}$$

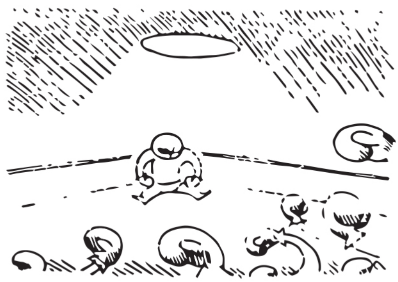
Question 2.2 *Show that every rational number $r \in \mathbb{Q}$ can be written as a finite continued fraction.*

$$r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

Question 2.3 *Show that for any $a, b \in \mathbb{Z}$ with $b \neq 0$ there are $q, r \in \mathbb{Z}$ with $0 \leq |r| < b$ such that: $a = bq + r$.*

3 A Putnam Continued Fraction

Question 3.1 (Putnam B4 1990) *Express* $\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}$ *in the form* $(a + b\sqrt{c})/d$, *where* a, b, c, d *are integers.*



Problems worthy
of attack
prove their worth
by hitting back.

Piet Hein