1 Arithmetic Expressions

Question 1.1 (Putnam and Beyond, p89 #267)**.** With the aid of a calculator that can add, subtract, and determine the inverse of a nonzero number, show that you can find the product of any two real numbers. *Bonus: Can you do it with at most 20 operations?*

Question 1.2 (24 Puzzle)**.** Find an expression that equals 24 and uses each of 1, 4, 5, 6 exactly once. You may use the operators $+$, $-$, $\cdot$, $\div$ any number of times.

2 Recursively Defined Sets

Question 2.1 (Putnam 2012, B1)**.** Let S be a set of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is in S.

Question 2.2 (Putnam 2017, A1)**.** Let S be the smallest set of positive integers such that

(i) 2 is in S;

(ii) n is in S whenever n^2 is in S;

(iii) $(n + 5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

Question 2.3 (Course Notes for CSC B36, Thm 4.2)**.** Let $\mathcal{S}$ be a set, B be a subset of $\mathcal{S}$, and $f : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ be an operator on $\mathcal{S}$. Prove that there is a unique subset S of $\mathcal{S}$ such that:

(i) S contains B;

(ii) S is closed under f;

(iii) Any subset of $\mathcal{S}$ that contains B and is closed under f contains S.