Research Statement: Parker Glynn-Adey

My work is concerned with the quantitative geometry of generic Riemannnian manifolds. To use Misha Kapovich's phrase, it is "geometric geometry" and studies quantities like lengths and areas. The guiding principle of my work has been to strengthen analogies between the robust and well-developed theory of expander graphs, and the nascent theory of high dimensional expander complexes and manifolds.

Informally put, an expander is a graph that is simultaneously sparse and highly connected. Despite having few edges, expanders connect every pair of vertices by many short paths. In the decades since their discovery, they have had a broad and sweeping impact on discrete mathematics and computer science [7].

Recently, expanders have been a useful combinatorial tool for addressing questions in pure Riemannian geometry. Their extremal properties have been used to construct counterintuitive metrics on manifolds. The influence of expander complexes on geometry will only increase as our understanding of the high dimensional analogues of expander graphs improves.

In my thesis, I built on the ideas that expander graphs are difficult to bisect and have large spectra, generalizing some of their behaviour to high dimensional manifolds. The main themes are isoperimetry, spectral estimates, and geometric subdivision problems. As a rule, the theorems are geometric but the proofs are topological or analytic.

Outline: The four sections below outline my work, and its relevant background material.

§1 **Background** contains the standard definition of expander graphs, the Cheeger inequality for graphs and manifolds, and the definition of width. There is a brief discussion motivating the heuristic that width is a non-linear analogue of the spectrum of a manifold.

§2 **Subdividing disks** is about my work on geometric subdivision problems in Riemannian 3-disks. This section includes joint work with Zhifei Zhu on knot theoretic links as metric obstructions to small width [4]. This work answered a question posed by Papasoglu [11].

§3 **Width-Volume Inequalities** describes my work on width-volume inequalities for Riemannian manifolds. This section includes joint work with Yevgeny Liokumovich on the relationship between width and Ricci curvature [3].

§4 **Sponges and Width** details exploratory work on Larry Guth's sponge problem. A positive answer to this problem would provide a novel and deep proof of the Euclidean width-volume inequality. In my thesis I showed that a related problem is solvable in the plane, and gave a new proof of the planar width volume inequality for nice sets. My work on the sponge problem also showed that the problem of deciding when there is an expanding embedding between two planar domains is NP-complete.

1 Background

In this section we give definitions of expander graphs and width. We will need these concepts for the other sections below.

We say that a graph $G = (V, E)$ is a λ-expander if: for every set S of vertices satisfying $|S| < |V|/2$ there are at least $\lambda|S|$ edges from S to $V \setminus S$. This definition makes clear the sense that expanders are highly connected: In order to separate any fixed percent of the vertices of the graph, one must cut $O(|V|)$ many edges of the graph. The difficulty of bisecting expanders has many important consequences.

In what follows, we will discuss the spectral approach to expansion. The connection between geometric subdivision problems and spectra is explained by Cheeger's inequality which holds both for Riemannian manifolds and graphs. We give the statement for manifolds,

Theorem 1.1 (Cheeger)**.** *For a compact Riemannian manifold (M, g) define:*

$$h(M) = \inf_H \frac{\text{area}(H)}{\min\{\text{vol}(A), \text{vol}(B)\}}$$

where the infimum is taken over all hypersurfaces H in M which subdivide M in to two disjoint submanifolds A and B. Let λ_1 be the smallest positive eigenvalue of the Laplacian of M. One has $\lambda_1 \geq h^2$.

Cheeger's inquality says that any manifold which requires a lot of area to bisect must have a large first eigenvalue. In applications, this spectral estimate has strong consequences: for example, the rate of convergence of a random walk on a graph to its stationary distribution is proportional to the first eigenvalue. One may say that expanders are robustly connected in the sense that any random walk visits everywhere quickly.

In §3 we will discuss width, a parametric or continuous notion related to the Cheeger constant on M. When estimating the Cheeger constant of a manifold, one looks at various different surfaces subdividing M. To estimate width, one looks at families of cycles which represent the global structure of M in certain sense described below — we say they sweep-out M. The global nature of a sweep-out makes it more robust and useful than the single slices used to estimate Cheeger constants. In particular, they can be used to estimate the size of minimal surfaces in a manifold.

We now introduce the theme of width; a measure of the complexity of slicing a manifold in to hypersurfaces. Let (M^n, g) be a compact Riemannian manifold. Using tools from geometric measure theory, one can metrize the space of k-cycles in M a natural way. A continuous $(n-k)$-dimensional family $z : X^{n-k} \to \mathcal{Z}_k(M)$ of k-cycles sweeps out M if z assembles to the fundamental class of M under a certain gluing operation. The precise statement of the gluing operation is somewhat technical, but one can imagine an $(n-k)$-dimensional family of k-cycles gluing together to form a n-chain.

Definition 1.1. The k-width of (M, g) is $\text{W}_k(M) = \inf_z \sup_p \text{vol}_k(z_p)$ where z ranges over all sweep-outs of M by k-cycles and the supremum runs over all cycles in the family.

An important example of a sweep out by $(n-1)$-cycles is obtained by taking the level sets of a smooth morse function $f : M \to \mathbb{R}$. One has that $f^{-1}(t)$ will be empty for $|t|$

sufficiently large. Thus, the family of $(n-1)$-cycles given by $z_t = f^{-1}(t)$ will start and end at $0 \in \mathcal{Z}_{n-1}(M)$. The important point about gluing together the level sets is that $f^{-1}([t-\epsilon, t+\epsilon])$ will be a n-chain of small volume with boundary $z_{t-\epsilon}$ and $z_{t+\epsilon}$ when ϵ is sufficiently small. Gluing the levels sets together, using the intermediate n-chains, gives a cycle representing the fundamental class of M.

2 Subdividing disks

The robust connectivity enjoyed by expanders motivated my work on disk subdivision. Before discussing my work on disk subdivision, we will introduce some of the history of the problem. Motivated by problems in geometric group theory [5], Gromov asked:

Question 2.1. *We say a Riemannian 2-disk is small if* $\text{length}(\partial D^2) \leq 1$ *and* $d(p, \partial D^2) \leq 1$ *for all* $p \in D^2$*. Is there a universal constant* C *such that the following holds? Every small 2-disk admits a homotopy of curves contracting its boundary circle to a point through curves of length at most* C*.*

At the time, Gromov was concerned with the diameter and area of van Kempen diagrams for groups. It was asked with the hope that a positive answer could reduce the growth rate of a bound on Dehn's function, a combinatorially defined isoperimetric profile for finitely presented groups.

S. Frankel and M. Katz [2] answered Gromov's question negatively by using a novel combinatorial construction. The key feature of their construction is the observation that the complete binary tree T_n has large combinatorial width: any continuous map $T_n \to \mathbb{R}$ must have a fiber containing $O(n)$ points. Their key observation was that any contraction of the boundary to a point would meet many edges of the tree. The metric they constructed was concentrated around the tree in such a way that any curve meeting it many times must be long. The use of combinatorics of trees to provide width estimates for spheres was further developed by Liokumovich [8]. Continuing in this vein, the work of Liokumovich, Nabutovsky, and Rotman [9] answered Gromov's question and subsequent questions raised by Frankel and Katz. Motivated by their work, Papasoglu asked in [11]:

Question 2.2. *Let* M *be a Riemannian manifold homeomorphic to a 3-disk satisfying: (i)* $\text{diam}(M) = d$*, (ii)* $\text{area}(\partial M) = A$*, (iii) and* $\text{vol}_3(M) = V$*. Is it true that there is a homotopy* $S_t : \partial M \times [0,1] \to M$ *such that:* $S_0 = id_{\partial M}$ *and* S_1 *is a point and* $\text{vol}_2(S_t) \leq f_1(A, d, V)$ *for some function* f_1*?*

Question 2.3. *Let* M *be as above. Is it true that there is a relative 2-disk* D *splitting* M *in to two regions of volume at least* $V/4$ *such that* $\text{area}(D) \leq f_2(A, d, V)$ *for some function* f_2*?*

In work with Zhifei Zhu [4], we answered Papasoglu's questions negatively. The construction involved linking a pair of tori in the disk. This linking construction is another example of the applicability of combinatorics to geometry. The kinds of obstructions that links can create in high dimensional metrics remains to be explored further. Independently, an elegant expander-based counterexample was given by Papasoglu and Swenson in [12].

In my thesis, I provide a positive answer related to Papasoglu's Question 2.3 about subdividing disks. We wish to partition a 3-sphere into two parts both of which contain at least a $1/4 - \epsilon$ fraction of the total volume. Any embedded surface which does so will be called a subdividing surface. We now introduce a quantity which computes the infimal area of a subdividing surface for M.

Definition 2.1. Given a Riemannian 3-sphere M with volume V, let

$$\mathrm{SA}_\epsilon(M) = \inf_{H \subset M} \left\{ \mathrm{vol}_2(H) : M \setminus H = X_1 \sqcup X_2, \mathrm{vol}_3(X_i) > \left(\frac{1}{4} - \epsilon\right) V \text{ for } i = 1, 2 \right\}$$

be the subdivision area of M. The infimum is taken over all embedded surfaces. We define

$$\mathrm{HF}_1(\ell) = \sup_{||z||_1 \leq \ell} \left(\inf_{\partial c = z} \mathrm{vol}_2(c) \right)$$

to be the first homological filling function. In the definition of $\mathrm{HF}_1(\ell)$ the supremum is taken over all 1-cycles z satisfying $\mathrm{vol}_1(z) \leq \ell$ and the infimum computes the size of the smallest 2-cycle c filling $z = \partial c$.

Homological filling functions were originally introduced by Gromov to study the large scale geometry of groups. They are a natural generalization of isoperimetric profiles to high co-dimension contexts. Whereas the isoperimetric profile quantifies the difficulty of filling a given amount of surface area by a volume, the k-th homological filling function quantifies the amount $(k+1)$-volume needed to fill a k-dimensional cycle. The homological filling functions were used by Nabutovksy and Rotman [10] to give the first curvature-free upper bound for the smallest area of a minimal hypersurface in a k-connected manifold.

And now we come back to disk subdivision. In my thesis, I showed:

Theorem 2.1 (Glynn-Adey & Zhu). *For any Riemannian 3-sphere (M, g) with M diffeomorphic to S^3 we have:*

$$\mathrm{SA}_\epsilon(M) \leq 3\,\mathrm{HF}_1(2d)$$

where d is the diameter of (M, g).

3 Width-Volume Inequalities

Before describing my work on width-volume inequalities, I recall a guiding principle of the work. The connection that I tried to draw out in my thesis is that width the non-linear analogue of (linear) spectral estimates for the Laplacian. Put simply the connection is as follows: The width of a manifold controls the size of the largest slice in an optimal decomposition of M by small k-slices. To compute $\mathrm{W}_{n-1}(M)$ one considers continuous families of slices. To estimate the smallest positive eigenvalue of M one may evaluate the Cheeger constant of M. The Cheeger constant is an infimum taken over single surfaces splitting M in to two parts. Width on the other hand, is properly considered a parametric version of this slicing. One looks for families of $(n-1)$-cycles in M with the hope of capturing more of the global geometry of M. Thus, width is a parametric or non-linear version of the

Cheeger constant $h(M)$. The parametric aspect of estimating width makes for a more robust measure of the expansion than the spectrum.

In [6], Guth proved the following remarkable width-volume inequality in Euclidean space:

Theorem 3.1. *If $U \subset \mathbb{R}^d$ is open and bounded then:*

$$\mathrm{W}_k(U) \leq C(d)\,\mathrm{vol}(U)^{\frac{k}{d}}$$

for a universal constant $C(d)$ depending only on dimension.

It is important to note that this inequality confirms our intuition: Euclidean objects should be easy to slice.ac On the other hand, the slices are necessarily non-linear. Besikovitch constructed an open bounded set in the plane with negligible measure which contains a segment of unit length in every direction. Guth's width-volume inequality shows that this set has small width, but the sweep-out witnessing this fact is subtle and non-linear.

The width-volume inequality does not hold for all Riemannian manifolds. The problem of characterizing exactly where it does hold remains wide open. The first positive result for a large class of manifolds was the following modified width-volume inequality:

Theorem 3.2 (Balacheff-Sabourau [1]). *If (Σ_k, g) is a closed Riemannian surface of genus k then:*

$$\mathrm{W}_1(\Sigma_k) \leq 10^{18}\sqrt{(k+1)\,\mathrm{area}(\Sigma_k)}$$

The dependence on genus and area is asymptotically optimal. Originally, arithmetic hyperbolic surfaces were used show the tightness of the dependence. One can also show tightness of the bound by using expanders to construct surfaces which are difficult to subdivide.

By using the uniformization theory for surfaces, I was able to improve the constant in Balacheff and Sabourau's result from 10^{18} to 220. This was done while attempting to generalize the width-volume inequality to higher dimensional manifolds. The main result of that work was:

Theorem 3.3 (Glynn-Adey & Liokumovich [3]). *If (M^n, g) is conformally non-negatively Ricci curved then:*

$$\mathrm{W}(M) \leq C(n)\,\mathrm{vol}(M)^{\frac{n-1}{n}}$$

4 Sponges and Width

In this last section we describe an approach to proving width-volume inequalities. Guth's width-volume inequality in Euclidean space would follow readily from a positive answer to the following:

Question 4.1 (Guth's Sponge Problem). *For any dimension d is there $\epsilon = \epsilon(d) > 0$ such that: If U is an open bounded set in the $\mathbb{R}^d$ of Lebesgue measure at most ϵ then there exists an embedding $U \to B^d(1)$ which increases the length of all curves.*

The intuition for the sponge problem is that a set of small measure which is large and diffuse should resemble a physical sponge. It has little volume itself, but encloses lot of volume in pockets which can be squeezed out. The problem however remains open, even in the plane. The algorithmic difficulty of the problem in general is illustrated by the following:

Proposition 4.1 (Glynn-Adey)**.** *Given U and V open bounded sets in the plane, it is NP-complete to determine if there is an expanding embedding from U to V.*

At present, the sponge problem in full generality seems out of reach. However, I was able to provide a positve solution to a related planar sponge problem. The result is sufficient to prove the width-volume inequality for Jordan measurable sets in the plane.

Theorem 4.2 (Glynn-Adey)**.** *If U is an open bounded Jordan measurable set in the $\mathbb{R}^2$ of measure one then there exists an embedding $U \to [0, 10] \times \mathbb{R}$ which increases the length of all curves.*

One might hope to explore the extremal sets for the three dimensional sponge problem by looking at the geometry of expander like subsets of $\mathbb{R}^3$. In the 1970s, motivated by speculation about the structure of the brain, Kolmogorov and Barzdin investigated 1-thick embeddings of graphs in to $\mathbb{R}^3$. Consider a graph as a topological simplicial complex where each edge is homeomorphic to an interval. An embedding $f : \Gamma \to \mathbb{R}^3$ is 1-thick if $d(f(\sigma_i), f(\sigma_j)) \geq 1$ for any non-adjacent simplices σ_i and σ_j.

Theorem 4.3 (Barzdin-Kolmogorov)**.** *If Γ is a graph with degree d with N vertices then Γ admits a 1-thick embedding in to $B^3(R)$ for $R \leq C(d)N^{1/2}$.*

The interesting part of this result is the exponent. Given that there are N vertices, and each vertex needs to be surrounded by a 1-ball, one might expect that that exponent ought to be 1/3. To show the sharpness of the exponent, Kolmogorov and Barzdin used random graphs and applied the expansion property of random graphs. Their proof is one of the earliest application of expansion to geometry.

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