The Classification Theorem of Surfaces

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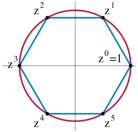
Classifying Mathematical Objects



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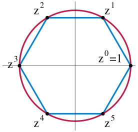

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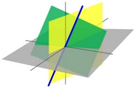

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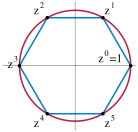

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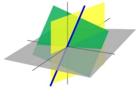

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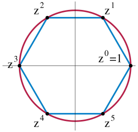

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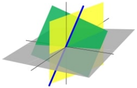

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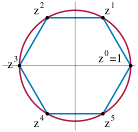

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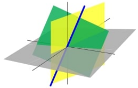

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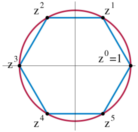

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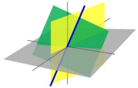

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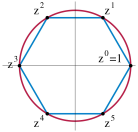

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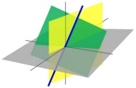

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For instance, an Isomorphism between Vector Spaces is a bijective Linear Transformation, since $T(\alpha v) = \alpha T(v)$ and $T(v + w) = T(v) + T(w)$ are properties that preserve vector addition and scalar multiplication, the structure of a Vector Space.

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Invariants

	e	$\mathbf{R_1}$	$\mathbf{R_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$
e	e	R_1	R_2	S_1	S_2	S_3
$\mathbf{R_1}$	R_1	R_2	e	S_3	S_1	S_2
$\mathbf{R_2}$	R_2	e	R_1	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3	e	R_1	R_2
$\mathbf{S_2}$	S_2	S_3	S_1	R_2	e	R_1
$\mathbf{S_3}$	S_3	S_1	S_2	R_1	R_2	e

Figure: The order of elements in a Group is an invariant under the group: they won't change under isomorphism

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$\mathbf{R_1}$	R_1	R_2	e	S_3	S_1	S_2
$\mathbf{R_2}$	R_2	e	R_1	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3	e	R_1	R_2
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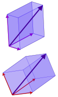

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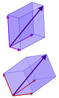

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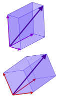

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R_1	R_1	R_2	e	S_3	S_1	S_2
R_2	R_2	e	R_1	S_2	S_3	S_1
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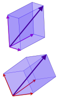

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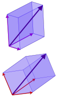

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An Invariant of a mathematical object A is a property of A that is unchanged under isomorphism.

In other words, if T is an isomorphism, then the property is true for both A and $T(A)$. This ensures objects are inequivalent if they do not share the property.

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Surfaces

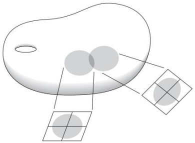

Figure: A surface "locally" looks like a subset of the plane

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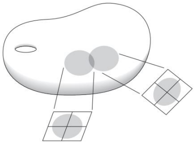

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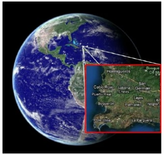

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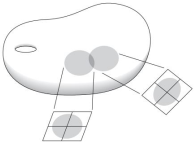

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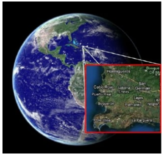

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We will define a Surface as a space that "locally" appears like a subset of the plane ($\mathbb{R}^2$).

By locally, I mean that around every point, there is a small region that is isomorphic to $U \subset \mathbb{R}^2$.

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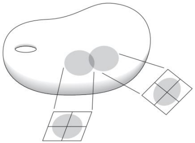

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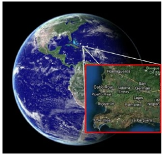

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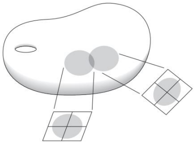

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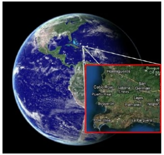

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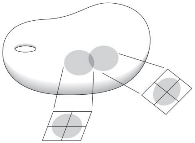

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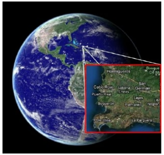

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Orientability and Non-Orientability

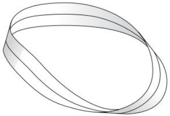
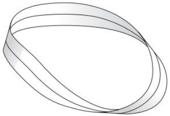


Figure: The Non-Orientable Mobius Strip. If you take a trip, you can go all around the surface. Spooky!

Orientability and Non-Orientability



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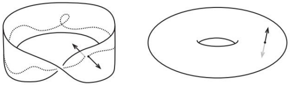

Figure: The Torus is orientable, the Mobius Strip is not.

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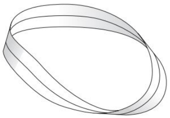

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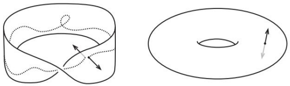

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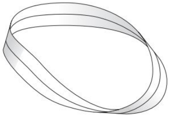

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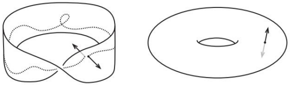

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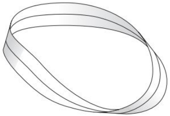

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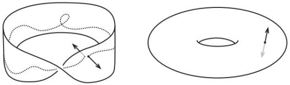

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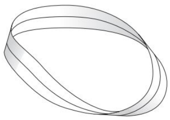

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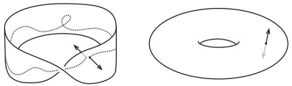

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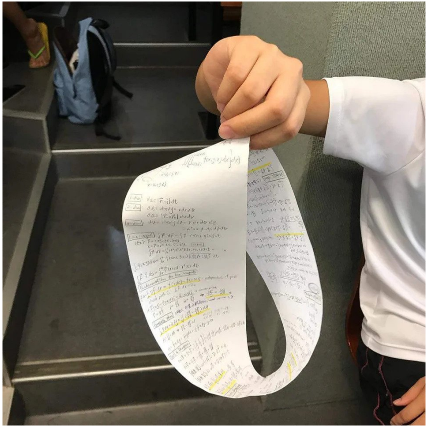

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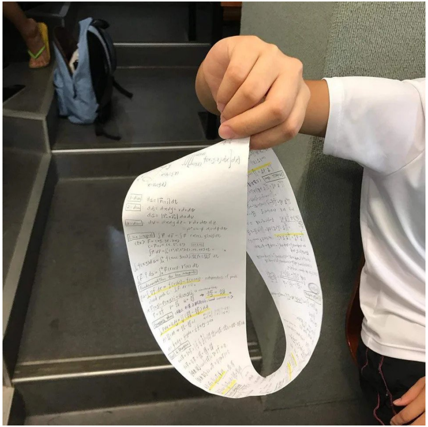

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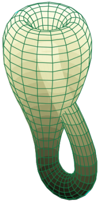

Figure: The Klein Bottle is a non-orientable surface that cannot be placed in $\mathbb{R}^3$

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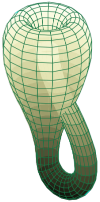

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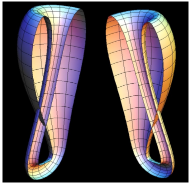

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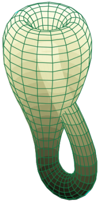

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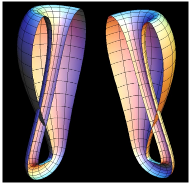

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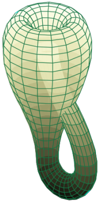

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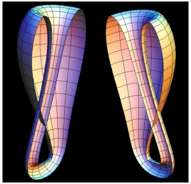

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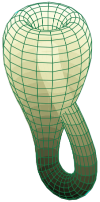

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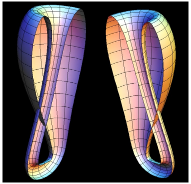

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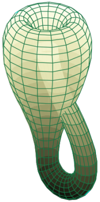

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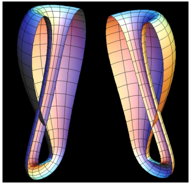

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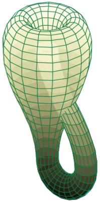

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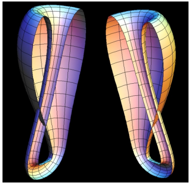

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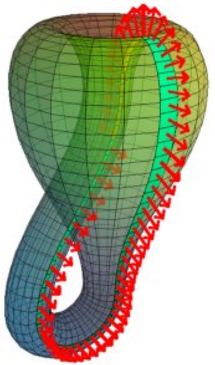

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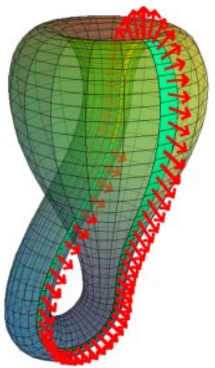

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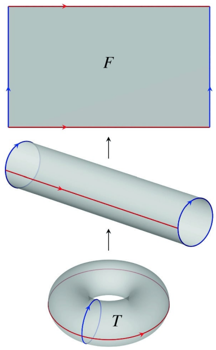

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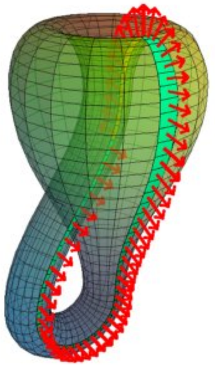

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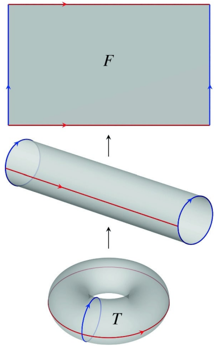

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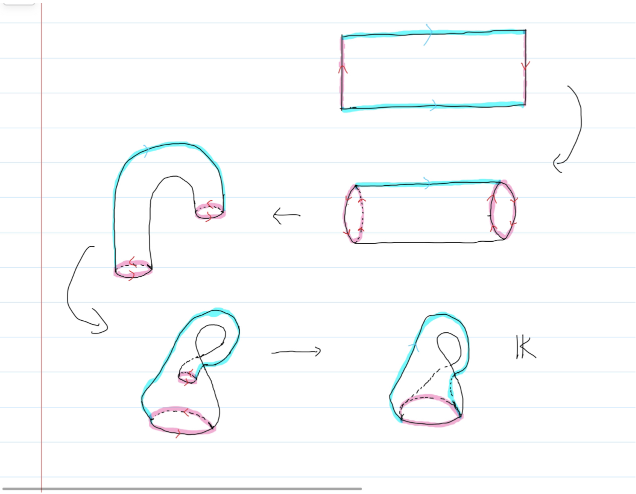

Figure: Pictoral representation of Gluing sides of a fund poly to make a Klein Bottle

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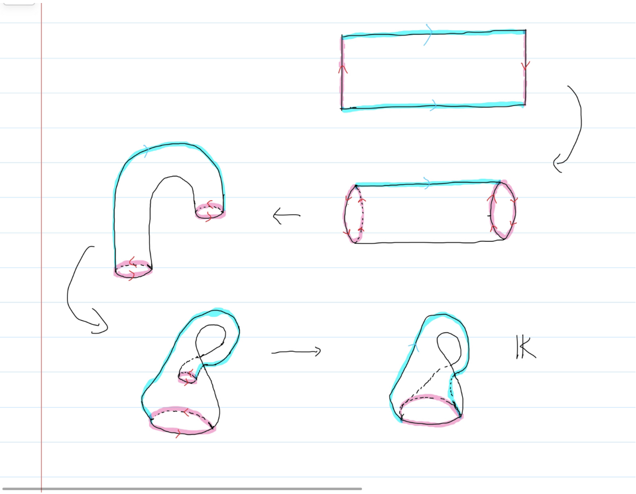

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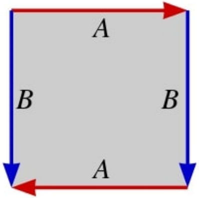
More Intrinsic/Polygon Represented Surfaces



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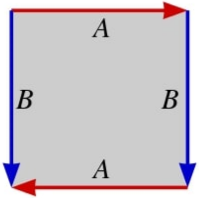

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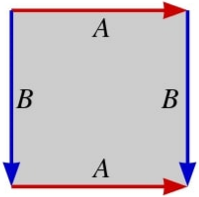

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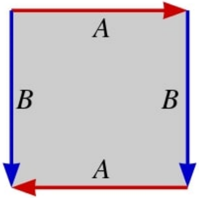

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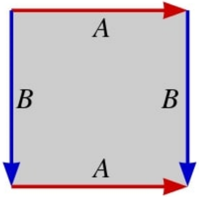

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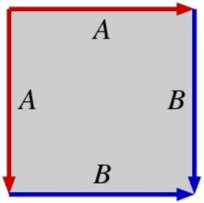

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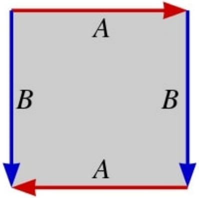

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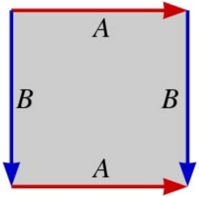

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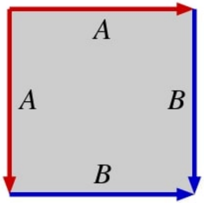

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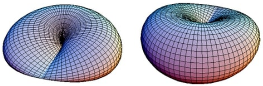

Figure: The Real Projective Plane in "Cross Cap" form: Note the self intersection required to view it in $\mathbb{R}^3$

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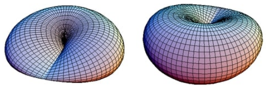

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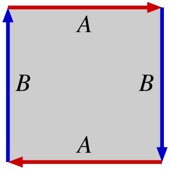

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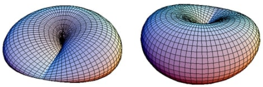

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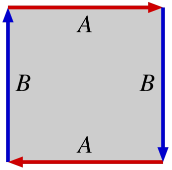

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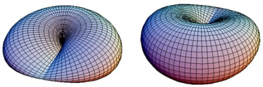

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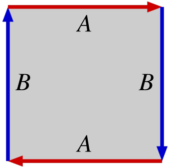

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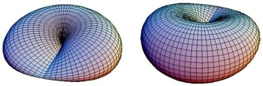

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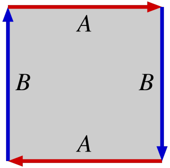

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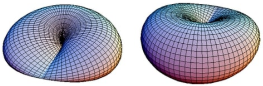

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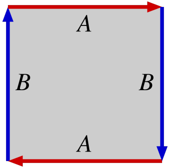

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The Projective Plane

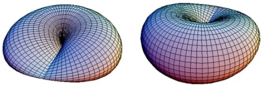

Figure: The Real Projective Plane in "Cross Cap" form: Note the self intersection required to view it in $\mathbb{R}^3$

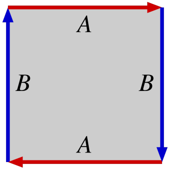

Figure: The Fundamental Polygon of $\mathbb{R}P^2$

Figure: $\mathbb{R}P^2$ can also be viewed as a semisphere glued to a Mobius Strip

- Another interesting example is the Projective Plane $\mathbb{R}P^2$. It was originally an object of study in Projective Geometry, but is also a Non-Orientable Surface.
- There are many ways of constructing this object: We can take the set of all lines through the origin of $\mathbb{R}^2$, glue the boundary lines of a Mobius strip to a semisphere, or glue diametrically opposite sides of a semisphere to each other.
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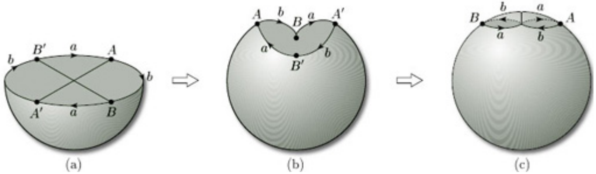

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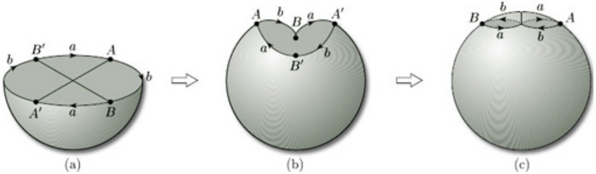

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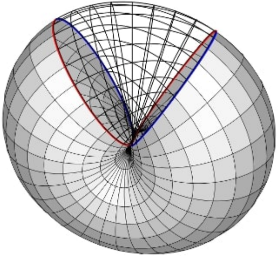

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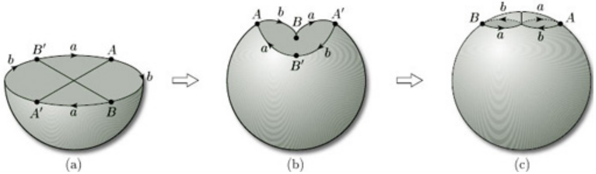

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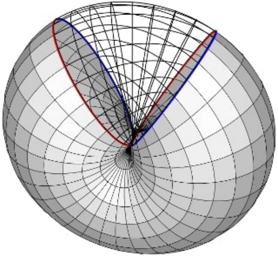

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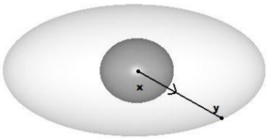

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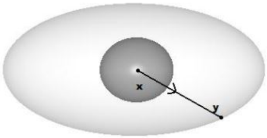

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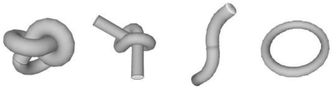

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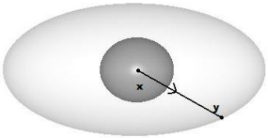

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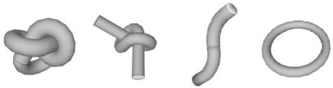

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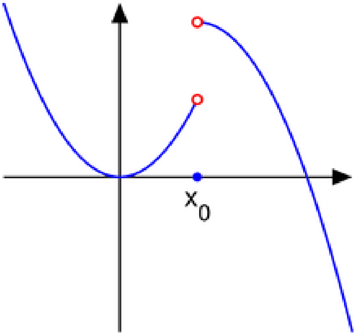

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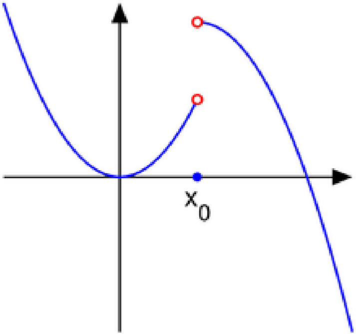

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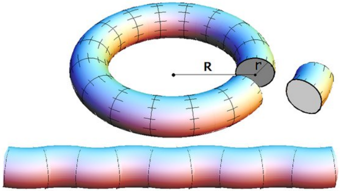

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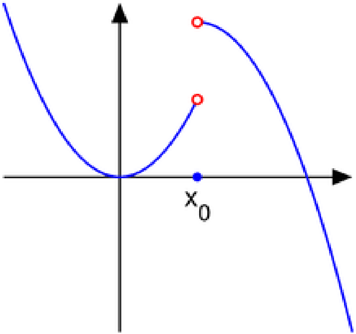

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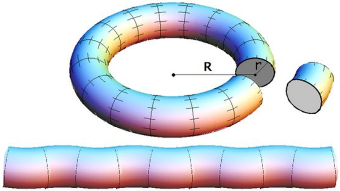

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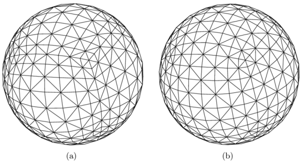

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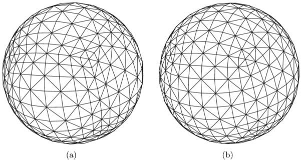

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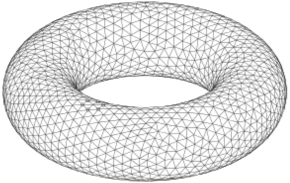

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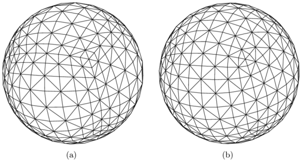

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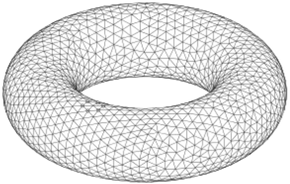

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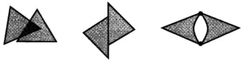

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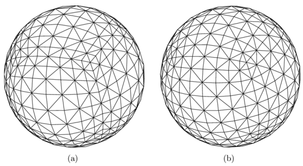

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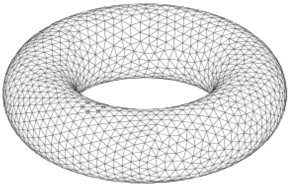

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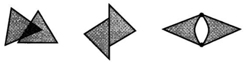

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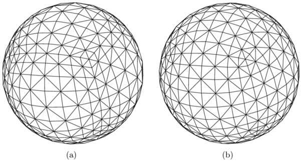

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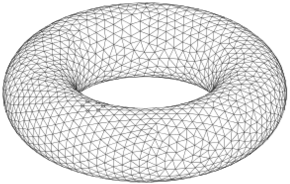

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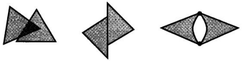

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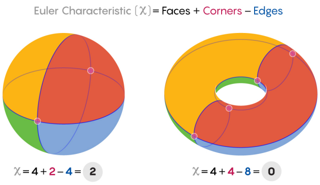

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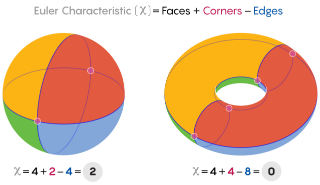

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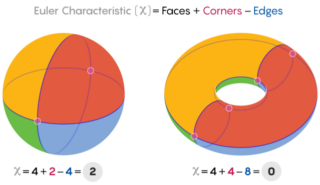

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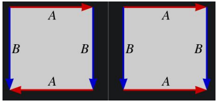

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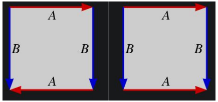

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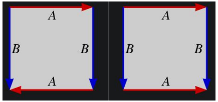

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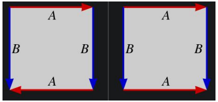

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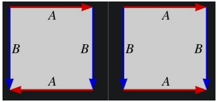

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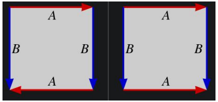

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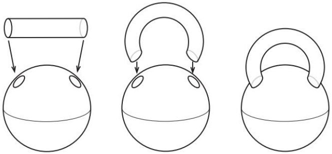

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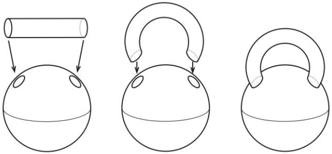

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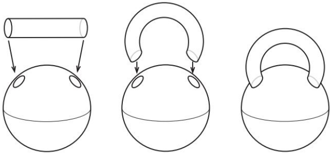

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- We will construct more Surfaces through a process called Surgery. This is gluing Surfaces we already know together to create new ones! It is in general, an important technique also used in other areas in Topology.

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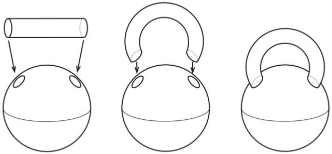

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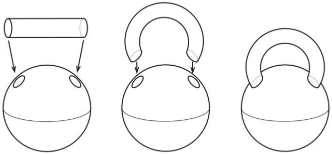

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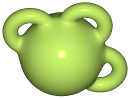

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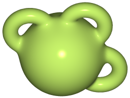

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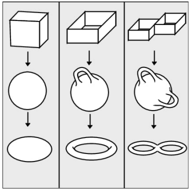

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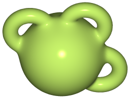

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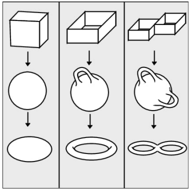

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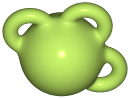

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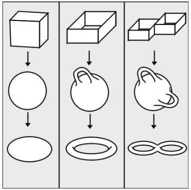

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Note: if $n = 0$, this is the sphere. If $n = 1$, this is a Torus. If $n = 2$, this is a two-holed Torus, and in general, we have something equivalent to an n holed Torus. Convince yourself of this. Note these are all topologically distinct.

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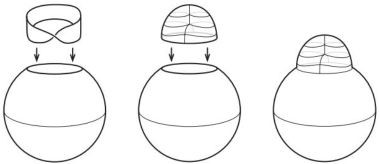

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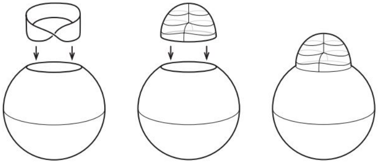

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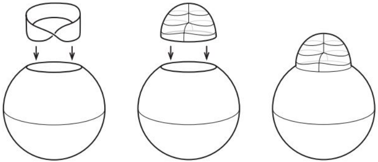

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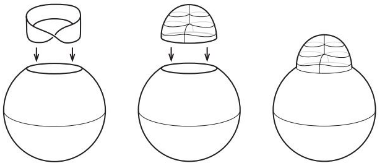

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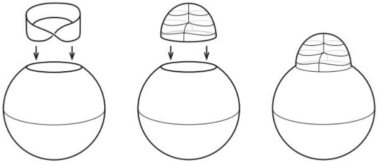

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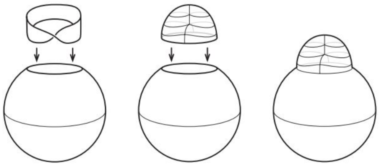

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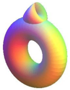

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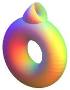

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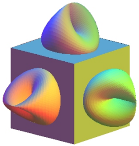

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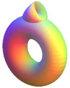

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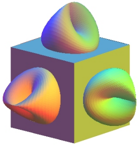

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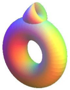

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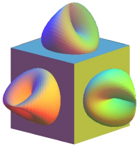

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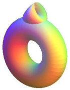

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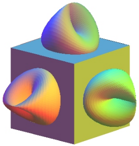

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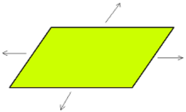

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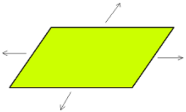

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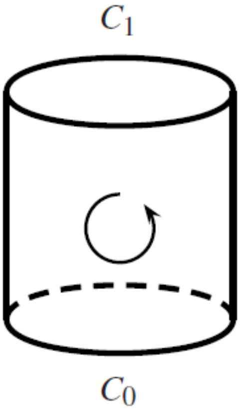

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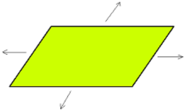

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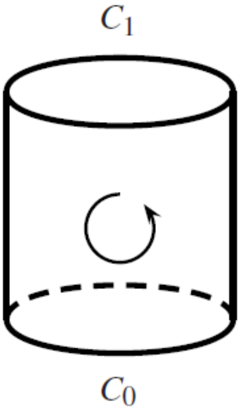

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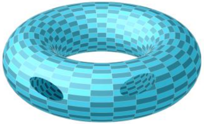

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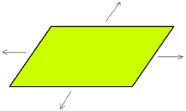

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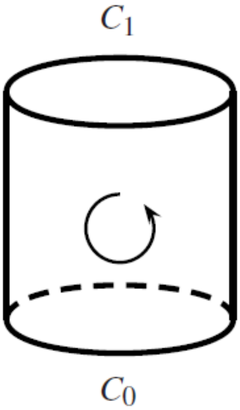

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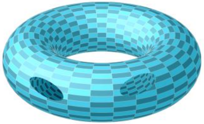

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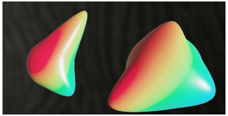

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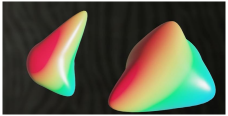

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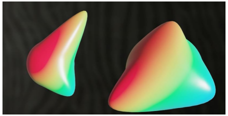

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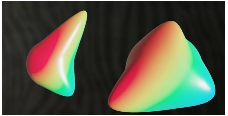

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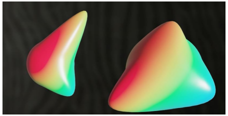

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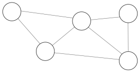

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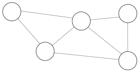
Graph Theory



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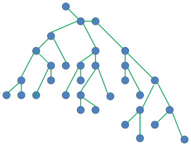

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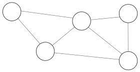

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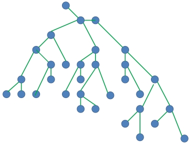

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To prove Lemma's 1 and 2, we must introduce some terminology and prove some results from Graph Theory.

Graph

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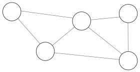
Graph Theory



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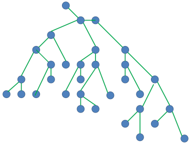

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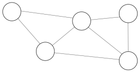

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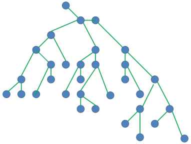

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We define the Euler Char of a Graph G to be
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Removing this edge gives us a graph of $e - 1$ edges, which by the IH has Euler char 1. Adding back the edge does not change the Euler char, so a graph with e edges must have an Euler char of 1. ■

Graph Theory

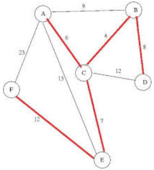
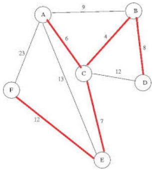


Figure: Turning a Graph L into a Tree by removing finitely many edges without disconnecting it.

Graph Theory



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If L is a graph containing a loop, then $\chi(L) < 1$.

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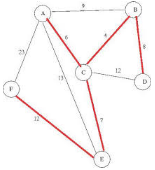

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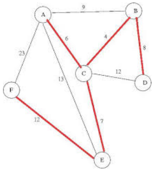

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This new graph, L', is a tree and we will have $\chi(L) = \chi(L') - g = 1 - g < 1$, as wanted ■

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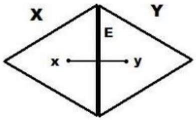

Figure 14:

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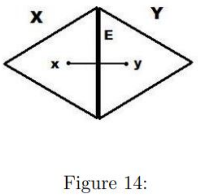
Graph Theory



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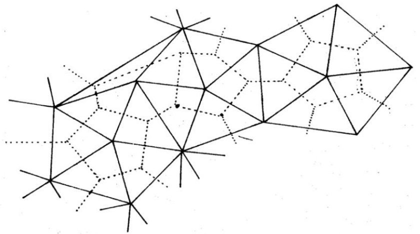

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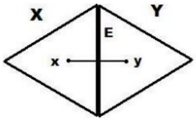
Graph Theory



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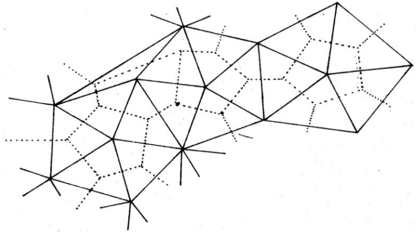

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The vertices and edges of a compliment K of a Dual Tree T form a connected graph.

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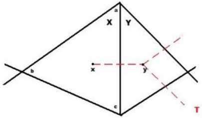

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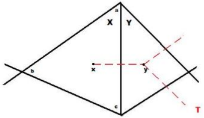

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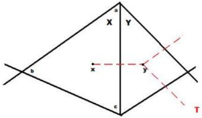

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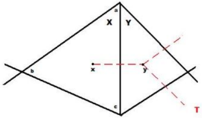

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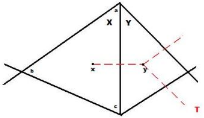

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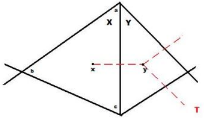

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Thus, K is connected, as wanted! ■

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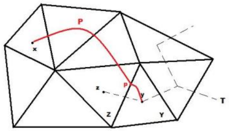

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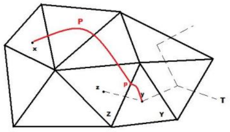

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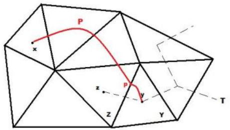

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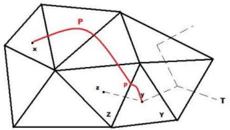

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Now, with these 2 Lemmas (Lemma 6 and 7) about Dual Tree's proven, we can finally proceed with the proofs of Lemma 1 and 2!

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Lemma 2

If S is a compact connected Surface without boundary, then we have the following are equivalent:

1. S is spherelike
2. $\chi(S) = 2$
3. S is homeomorphic to the Sphere

We will prove this by proving the chain of implications:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$$

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Letting $V, E, F, V_1, E_1, V_2, E_2$ be the vertices, edges, and faces in M, T, C resp, we have $V = V_2$, $F = V_1$, and $E = E_1 + E_2$. Then we have

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Since by Lemma 3, T will always have an end dual vertex, we can continously remove edges without disconnecting it.

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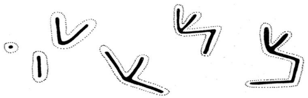

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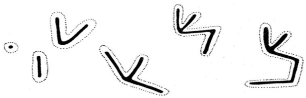

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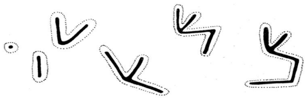

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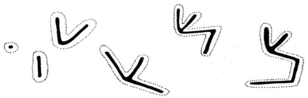

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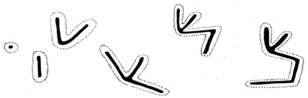

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So S is 2 disks glued along their boundary, i.e a Sphere, as wanted ■

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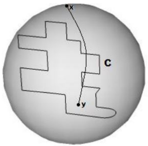

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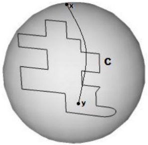

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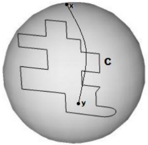

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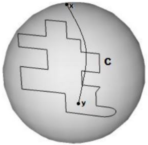

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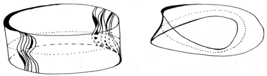

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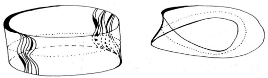

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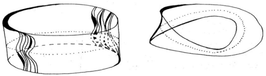

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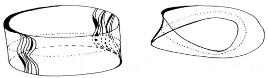

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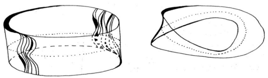

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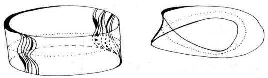

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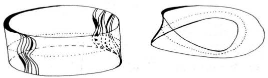

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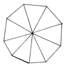

Figure: Gluing a Disk equivalent to joining vertices of C to a point

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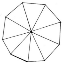

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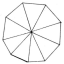

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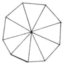

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Note that by Lemma 2, S_k is homeo to a sphere.

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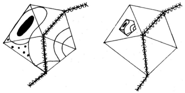

Figure: Shrinking a Disk D into its interior.

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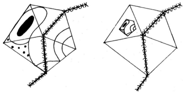

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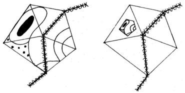

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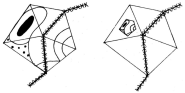

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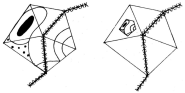

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There are 3 main types of desurgery:

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3. We have 1 disk left over. We simply glue a Mobius strip onto the boundary.

Proof

Figure: A Type One DeSurgery

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Figure: A Type Two DeSurgery

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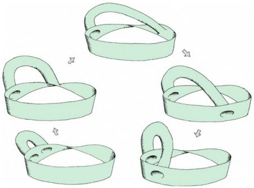

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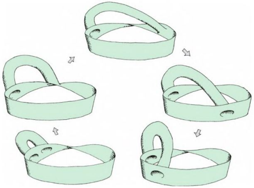

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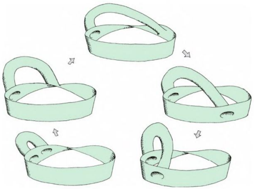

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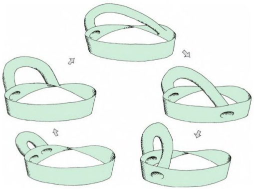

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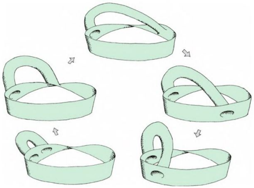

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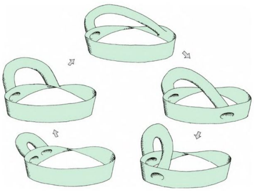

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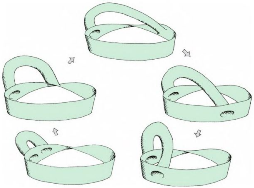

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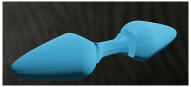

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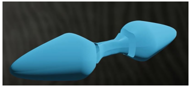

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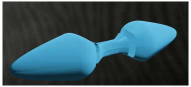

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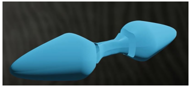

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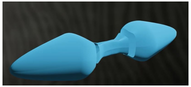

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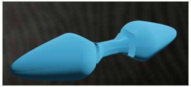

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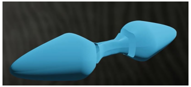

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Suggested Reading

- Textbooks:
- Topological Manifolds by John Lee (Will walk you through all the rigorous Topology you need to know for further study of differential Topology, motivated heavily by Manifolds/Surfaces and very geometric.
- Topology by Munkres (Another option for an introduction to Topology, a different approach to the subject then Lee)
- Other Books:
- Euler's Gem by David Richeson (A fantastic introduction to the history and motivation behind Topology at a beginner level)
- The Princeton Companion to Mathematics (A fantastic encyclopedia of Mathematics that has info on Topology and many other amazing fields of mathematics)
- Jeffery Weeks "The Shape of Space" (An awesome book that covers not only Classification of Surfaces but also 3-Manifolds and Geometry of Surfaces!)