The Classification Theorem of Surfaces

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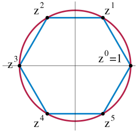
Classifying Mathematical Objects



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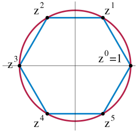

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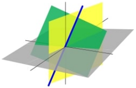

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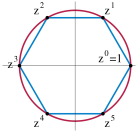

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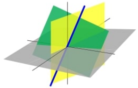

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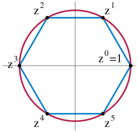

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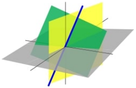

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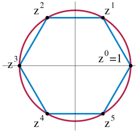

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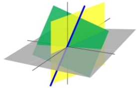

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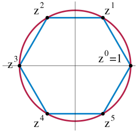

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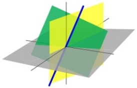

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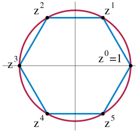

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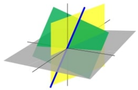

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For instance, an Isomorphism between Vector Spaces is a bijective Linear Transformation, since $T(\ \ v) = \ \ T(v)$ and $T(v + w) = T(v) + T(w)$ are properties that preserve vector addition and scalar multiplication, the structure of a Vector Space.

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Invariants

	$\mathbf{e}$	$\mathbf{R_1}$	$\mathbf{R_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$
$\mathbf{e}$	e	R_1	R_2	S_1	S_2	S_3
$\mathbf{R_1}$	R_1	R_2	e	S_3	S_1	S_2
$\mathbf{R_2}$	R_2	e	R_1	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3	e	R_1	R_2
$\mathbf{S_2}$	S_2	S_3	S_1	R_2	e	R_1
$\mathbf{S_3}$	S_3	S_1	S_2	R_1	R_2	e

Figure: The order of elements in a Group is an invariant under the group: they won't change under isomorphism

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	e	R_1	R_2	S_1	S_2	S_3
e	e	R_1	R_2	S_1	S_2	S_3
R_1	R_1	R_2	e	S_3	S_1	S_2
R_2	R_2	e	R_1	S_2	S_3	S_1
S_1	S_1	S_2	S_3	e	R_1	R_2
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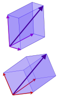

Figure: Number of basis elements is an invariant of a Vector Space

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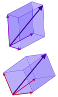

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S_1	S_1	S_2	S_3	e	R_1	R_2
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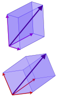

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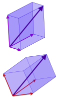

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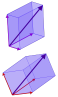

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Invariants

An Invariant of a mathematical object A is a property of A that is unchanged under isomorphism.

In other words, if T is an isomorphism, then the property is true for both A and $T(A)$. This ensures objects are inequivalent if they do not share the property.

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Surfaces

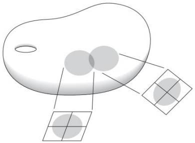

Figure: A surface "locally" looks like a subset of the plane

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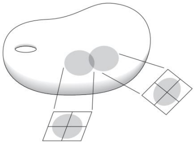

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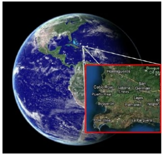

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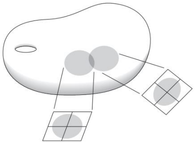

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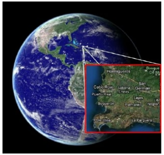

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We will define a Surface as a space that "locally" appears like a subset of the plane ($\mathbb{R}^2$).

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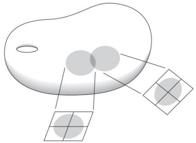

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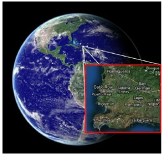

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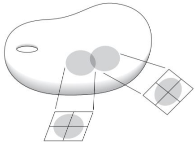

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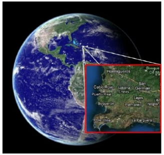

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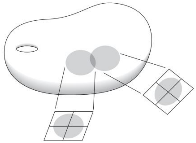

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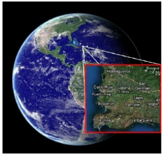

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- Not all surfaces are as nice as the Sphere or Torus though!
- Some are one-sided or more precisely Non-Orientable

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Note: if $n = 0$, this is the sphere. If $n = 1$, this is a Torus. If $n = 2$, this is a two-holed Torus, and in general, we have something equivalent to n-holed Torus. Convince yourself of this. Note these are all topologically distinct.

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Trees

Graphs may have loops between vertices. A graph that contains no loops is called a tree.

Graph Theory

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We de ne the Euler Char of a Graph G to be

$$(G) = v \quad e$$

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Removing this edge gives us a graph of $e \quad 1$ edges, which by the IH has Euler char 1. Adding back the edge does not change the Euler char, so a graph with e edges must have an Euler char of 1.

Graph Theory

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This new graph, L^0, is a tree and we will have $(L) = \quad (L^0) \quad g = 1 \quad g < 1$, as wanted

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The vertices and edges of a compliment K of a Dual Tree T form a connected graph.

Proof: Note that K contains all the vertices in the Triangulation Q, so it su ces to prove that any two of these vertices can be joined by a path of edges in K.

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Thus, K is connected, as wanted!

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Now, with these 2 Lemmas (Lemma 6 and 7) about Dual Tree's proven, we can finally proceed with the proofs of Lemma 1 and 2!

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$$(S) = v \quad e + f = v_2 \quad e_1 \quad e_2 + v_1$$

Rearranging, we get

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$$(S) = \quad (T) + \quad (C) = 1 + \quad (C) \quad 1 + 1 = 2 \text{ as wanted!}$$

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Proof of Lemma 2

Recall:

Lemma 2

If S is a compact connected Surface without boundary, then we have the following are equivalent:

1. S is spherelike
2. (S) = 2
3. S is homeomorphic to the Sphere

We will prove this by proving the chain of implications:

$$(1) \;) \quad (2) \;) \quad (3) \;) \quad (1)$$

Proof of Lemma 2

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Let S be a Surface in the above sense, and assume S is spherelike but $(S) \neq 2$ for a contradiction.

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Letting $V; E; F; V_1; E_1; V_2; E_2$ be the vertices, edges, and faces in $M; T; C$ resp, we have $V = V_2$, $F = V_1$, and $E = E_1 + E_2$. Then we have

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$$(S) = V \quad E + F = V_2 \quad E_1 \quad E_2 + V_1 = \quad (T) + \quad (C)$$

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By Lemma 6, T contains all the dual vertices in S, and as T is a Tree and therefore a connected set of vertices and edges, we can take a path on T from one disjoint component to another that does not meet C and so does not meet the loop. (because T will never meet C by defn of C)

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Since by Lemma 3, T will always have an end dual vertex, we can continously remove edges without disconnecting it.

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Now, consider a small neighbohrhood around this vertex. This is homeo to a disk. Grow back out each edge and vertex, extending this small neighbohrhood to cover each edge.

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So S is 2 disks glued along their boundary, i.e a Sphere, as wanted

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With this in mind, we say that a point y on the Sphere has even if the arc xy has even parity, and odd otherwise.

Along any path not containing C, the parity remains constant (even) so C divides S into 2 distinct set of points: even and odd, as wanted.

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There are 3 main types of desurgery:

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3. We have 1 disk left over. We simply glue a Mobius strip onto the boundary.

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Then S^0 cannot have any Mobius Strips or Klein Bottles in it, so only desurgeries of type 1 can occur. This means that $S = S^0$ is a Sphere with m handles sewn on.

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Suggested Reading

- Textbooks:
- Topological Manifolds by John Lee (Will walk you through all the rigorous Topology you need to know for further study of differential Topology, motivated heavily by Manifolds/Surfaces and very geometric.
- Topology by Munkres (Another option for an introduction to Topology, a different approach to the subject then Lee)
- Other Books:
- Euler's Gem by David Richeson (A fantastic introduction to the history and motivation behind Topology at a beginner level)
- The Princeton Companion to Mathematics (A fantastic encyclopedia of Mathematics that has info on Topology and many other amazing fields of mathematics)
- Jeffery Weeks "The Shape of Space" (An awesome book that covers not only Classification of Surfaces but also 3-Manifolds and Geometry of Surfaces!)