

An Introduction to Descriptive Set Theory

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Preliminary notions

For the following definitions, we let X be a set equipped with a topology τ .

Definition

We say that $D \subseteq X$ is **dense** in X if $D \cap U \neq \emptyset$ for any $U \in \tau$.

Definition

We say that X is **separable** if it contains a countable dense set. For example, \mathbb{R} with its usual topology is separable since \mathbb{Q} is dense in \mathbb{R} .

Preliminary notions cont.

Definition

We say that X is **metrizable** if there exists a metric d on X such that τ is equal to the topology induced by d .

Definition

We say that (X, d) is **complete** if every Cauchy sequence in X converges to a point in X .

Preliminary notions cont.

Definition

We say that Y is G_δ in X if $Y = \bigcap_{n \in \mathbb{N}} U_n$ where $U_n \in \tau$. We say that Y is F_σ in X if $Y = \bigcup_{n \in \mathbb{N}} U_n^c$ where $U_n \in \tau$. Note that the complement of a G_δ set is F_σ and vice versa.

Preliminary notions cont.

Definition

Let $\{X_i\}_{i \in I}$ be a collection of topological spaces and define $X := \prod_{i \in I} X_i$. The **product topology** on X is the smallest topology on X such that all projection maps are continuous.

Basic open sets in X are of the form $\prod_{i \in I} U_i$ where U_i is open in X_i and $U_i = X_i$ for all but finitely many U_i .

What is a Polish space?

Definition

A **Polish space** is a separable, completely metrizable topological space.

Remark

Note that Polish spaces are *completely metrizable*, not complete with respect to a specific metric: i.e $(0, 1)$ as a subspace of \mathbb{R} with the usual topology is not complete, but there exists a complete metric on $(0, 1)$ (ex. any homeomorphism from $(0, 1)$ to \mathbb{R} induces a complete metric on $(0, 1)$).

Examples of Polish spaces

Some simple examples of Polish spaces:

- \mathbb{R}^n
- $\mathbb{R}^{\mathbb{N}}$
- \mathbb{C}^n
- $\mathbb{C}^{\mathbb{N}}$
- The n -dimensional cube
- Any countable set with the discrete topology

Special Examples of Polish spaces

Definition

The Cantor space, denoted \mathcal{C} , is defined by $\mathcal{C} := \{0, 1\}^{\mathbb{N}}$ equipped with the product topology.

Definition

The Hilbert cube, denoted $\mathbb{I}^{\mathbb{N}}$, is defined by $\mathbb{I}^{\mathbb{N}} := [0, 1]^{\mathbb{N}}$ equipped with the product topology.

Definition

The Baire space, denoted \mathcal{N} , is defined by $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ equipped with the product topology.

Polish subspaces of Polish spaces

Theorem

If X is metrizable and $Y \subseteq X$ is completely metrizable, then Y is G_δ in X . Conversely, if X is completely metrizable and $Y \subseteq X$ is G_δ , then Y is completely metrizable. In particular, a subspace Y of a Polish space X is Polish iff Y is G_δ in X .

Polish spaces and the Hilbert cube

Theorem

Every separable metrizable space is homeomorphic to a subspace of $\mathbb{I}^{\mathbb{N}}$. In particular, Polish spaces are, up to homeomorphism, G_{δ} subspaces of the Hilbert cube.

Some definitions

Definition

A point x in a topological space is an **isolated point** if $\{x\}$ is open.

Definition

A point x in a topological space is a **limit point** if it isn't an isolated point.

Definition

A space is **perfect** if it contains no isolated points.

Definition

A point x in a topological space is a **condensation point** if every open neighbourhood of x is uncountable.

Some definitions cont.

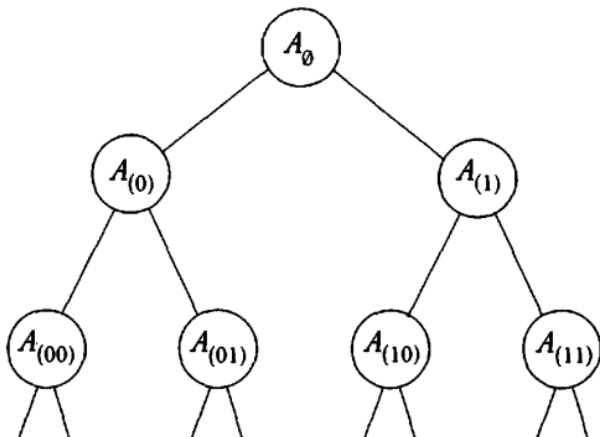
We can think of \mathcal{C} as a tree, which motivates the following definition:

Definition

A **Cantor scheme** on a set X is a collection $(A_s)_{s \in 2^{<\mathbb{N}}}$ of subsets of X satisfying:

- $A_{s \smallfrown 0} \cap A_{s \smallfrown 1} = \emptyset$ for $s \in 2^{<\mathbb{N}}$
- $A_{s \smallfrown i} \subseteq A_s$ for $s \in 2^{<\mathbb{N}}, i \in \{0, 1\}$

Some definitions cont.



Embedding the Cantor space

Theorem

Let X be a non-empty, perfect Polish space. Then there is an embedding of \mathcal{C} into X .

The Cantor-Bendixson theorem

Cantor-Bendixson Theorem

If X is a Polish space, then X can be uniquely written as $X = P \cup C$ where P is perfect in X and C is open and countable.

Corollary

Any uncountable Polish space contains a copy of \mathcal{C} and thus has cardinality \mathfrak{c} .

Borel sets

Definition

Given a set X , we say that $M \subseteq P(X)$ is a σ -algebra on X if M is closed under countable unions and complements.

Given $A \subseteq P(X)$, we refer to the smallest σ -algebra containing A as the σ -algebra **generated by** A .

Definition

The class of **Borel sets** of a topological space X is the σ -algebra generated by the open sets of X .

Borel sets (cont.)

Theorem

Let (X, τ) be Polish and let $A \subseteq X$ be Borel. Then there is a Polish topology $\tau_A \supseteq \tau$ such that τ and τ_A generate the same Borel sets and A is clopen in τ_A .

The perfect set theorem for Borel sets

Perfect set theorem for Borel sets

Let (X, τ) be Polish and $A \subseteq X$ be uncountable and Borel. Then A contains a copy of \mathcal{C} .

Proof: We extend τ to τ_A in which A is clopen and Borel sets of τ_A are Borel sets of τ . Since A is closed in τ_A , it's G_δ and therefore Polish. Since A is uncountable and Polish, by Cantor-Bendixson A contains a copy of \mathcal{C} .

Null and meager sets

Definition

Given $A \subseteq \mathbb{R}$, we say that A has **Lebesgue measure 0** or that A is a **null set** if, for any $\epsilon > 0$, A can be covered by some $\{I_n\}_{n \in \mathbb{N}}$ where I_n is an open interval and $\sum_{n \in \mathbb{N}} \text{length}(I_n) < \epsilon$.

Definition

Given $U \subseteq \mathbb{R}$, we say that U is **nowhere dense** in \mathbb{R} if the closure of U has empty interior. We say that $A \subseteq \mathbb{R}$ is **meager** if $A = \bigcup_{n \in \mathbb{N}} U_n$ where each U_n is nowhere dense in \mathbb{R} .

Baire category

Remark

If $A \subseteq \mathbb{R}$ is meager, it is said to be of **first category**. If A is non-meager, it is said to be of **second category**. We will avoid this convention to prevent confusion.

The Baire category theorem

Baire category theorem

If X is completely metrizable, then it satisfies the following equivalent conditions:

- Every non-empty open subset of X is non-meager.
- Every comeager set in X is dense in X .
- The intersection of countably many dense open sets in X is dense.

Proof: Let $\{U_n\}_{n \in \mathbb{N}}$ be dense open and let U be open in X . We will show that $\bigcap_{n \in \mathbb{N}} U_n \cap U \neq \emptyset$.

Note that $U_1 \cap U \neq \emptyset$ so construct a sufficiently small $B_1 \subseteq U_1 \cap U$ such that $\overline{B_1} \subseteq U_1 \cap U$. Then $\overline{B_1} \cap U_2 \neq \emptyset$ so construct a smaller $B_2 \subseteq U_2 \cap B_1$ such that $\overline{B_2} \subseteq B_1 \cap U_2$. Repeat this process inductively.

Let x_i be the centre of B_i . Then (x_i) is a Cauchy sequence and by completeness $x_i \rightarrow x \in \bigcap_{n \in \mathbb{N}} B_n \subseteq \bigcap_{n \in \mathbb{N}} U_n \cap U$.

The continuum hypothesis

Definition

We define $\aleph_0 := |\mathbb{N}|$; a set S is **countable** if $|S| \leq \aleph_0$ and **uncountable** otherwise. We denote the first uncountable ordinal by \aleph_1 .

The continuum hypothesis

The continuum hypothesis (denoted CH) is the statement that $\mathfrak{c} = \aleph_1$, or equivalently that if $A \subseteq \mathbb{R}$ is uncountable then $|A| = \mathfrak{c}$.

It has been shown that CH is independent of ZFC, meaning that we can freely assume either CH or \neg CH without contradiction.

Duality of measure and category

Assume CH. Then, if φ is a statement about subsets of \mathbb{R} and φ^* is the same statement but with all occurrences of "Lebesgue measure 0" replaced with "meager" or vice versa, then $\varphi \iff \varphi^*$. This is thanks to the following result:

Erdős-Sierpiński theorem

Assume CH. Then there exists a bijection $f : \mathbb{R} \rightarrow \mathbb{R}$ such that, for any $A \subseteq \mathbb{R}$, A is null iff $f(A)$ is meager and vice versa.

Duality of measure and category cont.

Lemma

There exist disjoint $F, G \subseteq \mathbb{R}$ such that F is null, G is meager and $F \cup G = \mathbb{R}$.

Proof: Enumerate \mathbb{Q} by $\{q_i\}_{i \in \mathbb{N}}$ and for $n \in \mathbb{N}$ define $U_n := \bigcup_{i \in \mathbb{N}} (q_i - \frac{1}{2^{i+n}}, q_i + \frac{1}{2^{i+n}})$. Then $F := \bigcap_{n \in \mathbb{N}} U_n$ is null and $G := F^c$ is meager.

Duality of measure and category cont.

Lemma

If $F, G \subseteq \mathbb{R}$ are null and meager respectively, there exist respectively null and meager $F^+, G^+ \subseteq \mathbb{R}$ such that $F \subseteq F^+, G \subseteq G^+$ and $|F^+ \setminus F| = |G^+ \setminus G| = \mathfrak{c}$.

Proof (sketch): WLOG, G is F_σ . Then G^c is Polish and comeager, and therefore uncountable by the Baire category theorem, so it contains a copy of \mathcal{C} by Cantor-Bendixson, so define $G^+ := G \cup \mathcal{C}$.

WLOG, F is Borel, so F^c is Borel and uncountable, and so by the perfect set theorem it contains a copy of \mathcal{C} . Let $\{q_i\}_{i \in \mathbb{N}} \subseteq \mathcal{C}$ be dense and for $n \in \mathbb{N}$ define $U_n := \bigcup_{i \in \mathbb{N}} (q_i - \frac{1}{2^{i+n}}, q_i + \frac{1}{2^{i+n}})$. Then

$F^+ := F \cup \bigcap_{n \in \mathbb{N}} U_n$ is our desired null set.

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Thanks for your attention!