Modelling Mathematics with Knitting and Crochet

Kevin Santos

University of Toronto

16 March 2022

Outline

Table of Contents

Euclidean geometry

The 2D geometry we are most familiar with is Euclidean geometry. This geometry is based on a set of axioms/postulates originally established by Euclid, which describe the behaviour of lines and points.

1. We can draw a unique straight line between any two points.
2. We can extend any straight line indefinitely.
3. We can draw a circle with a specified centre and radius.
4. All right angles are equal to one another.
5. **Parallel postulate:** Given a line L and a point P outside of the line, there is a **unique** line that passes through P and is parallel to L.

Non-Euclidean geometry

Parallel postulate: Given a line L and a point P outside of the line, there is a **unique** line that passes through P and is parallel to L.

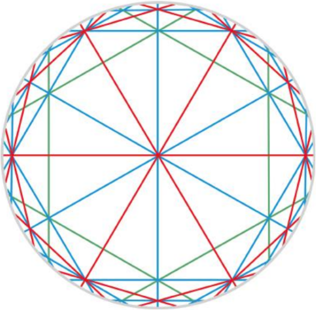
Altering the fifth axiom is one way to describe different **non-Euclidean geometries**, such as spherical and hyperbolic geometry.

Hyperbolic geometry

In hyperbolic geometry, given a straight line L and a point P not on L, there are **at least two distinct straight lines** through P that are parallel to L.

Geometric models

Mathematicians have modelled the hyperbolic plane by giving alternate descriptions of straight lines.



Source: [1]

The Beltrami model

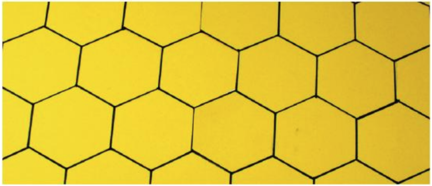
Surface curvature

We tend to think of Euclidean geometry as “living in” a flat plane. What kind of surface would have hyperbolic geometry?

Riemann proved that surfaces with **constant negative curvature** have **local hyperbolic geometry** [1].

Surface curvature

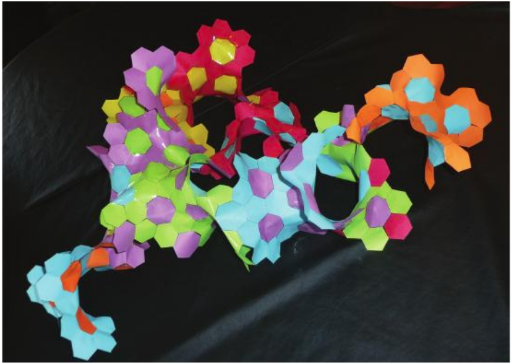
Surfaces with **positive curvature** curve "outward" and eventually "close up"; surfaces with **zero curvature** are flat. One way of visualizing curvature is to imagine tiling a surface with hexagons.



Source: [1]

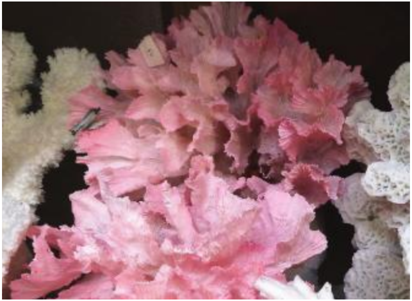
Surface curvature

Surfaces with negative curvature appear to fold into themselves and form ruffles, and they can be extended indefinitely.



Source: [1]

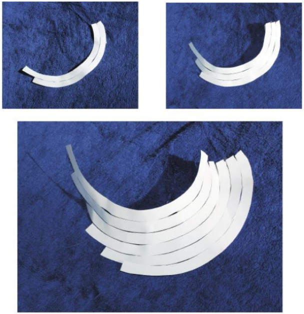
Negative curvature in nature



Source: [1]

Modelling the hyperbolic plane

William Thurston developed a model of the hyperbolic plane using paper annuli.



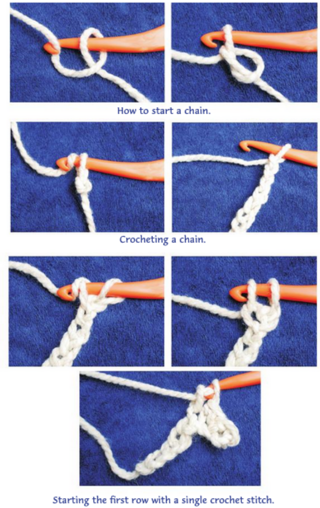
Source: [1]

Daina Taimina sought to use crochet to produce a similar model.

Source: [1]

How crochet works

Crocheted fabric begins with a series of loops of yarn. A crochet hook is used to pass loops of yarn through each other in a series of rows.



Source: [1]

Why crochet?

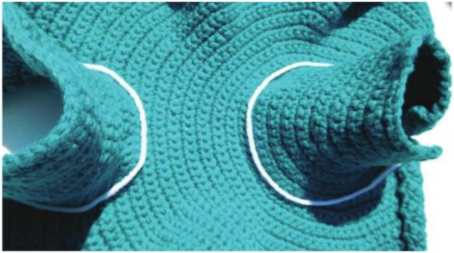
- Crocheted fabric is both sturdy and easy to manipulate.
- Crocheted fabric is built by working each stitch one at a time.

Modelling the hyperbolic plane

To model the hyperbolic plane, we start with a row and add an **increase every Nth stitch** for some N. i.e. make N stitches normally, then make the $N + 1$th stitch, then work a stitch in the same loop as the $N + 1$th stitch. Repeat until the end of the row, then turn the work and apply the same pattern.

Different radii

Choosing different values of N gives rise to hyperbolic planes with different radii.



Source: [1]

Doing geometry on the crochet model

Straight lines can be formed on the crochet model by folding it.

Source: [1]

Doing geometry on the crochet model

The crochet model can be used to visualize properties of hyperbolic geometry, such as the hyperbolic parallel postulate.

Source: [1]

Doing geometry on the crochet model

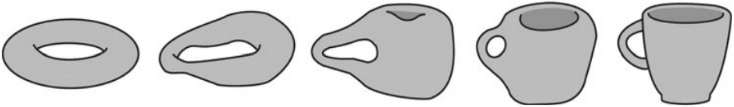
Triangles in the hyperbolic plane have angle sum $< 180^\circ$.

Source: [1]

Table of Contents

Classifying topological spaces

Topological spaces are said to be equivalent if they are **homeomorphic**; this means that one can be **continuously deformed** to form another. Topology is concerned with properties of spaces that are conserved under homeomorphisms.

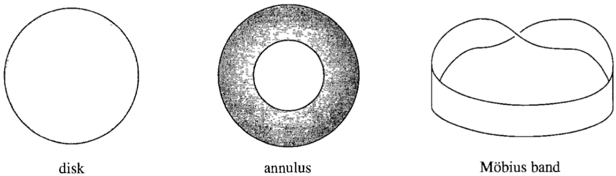


Source: `https://www-prod.media.mit.edu/publications/shape-changing-interfaces/`

A circular disk is equivalent to a square; a torus is equivalent to a coffee mug.

Topological surfaces

In topology, a surface is a smooth compact 2-manifold. We are mainly concerned with surfaces without boundary.
Roughly, a surface is a two-dimensional mathematical object of finite size with no edges, pinch-points, or sharp corners.
A 2-manifold is a space where every point has a neighborhood homeomorphic to $\mathbb{R}^2$.



Source: [2]

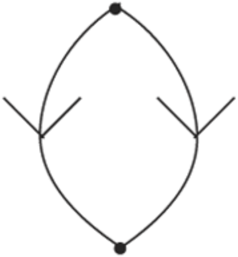
Disks, annuli, and Mobius bands have boundaries.

Examples of topological surfaces

The most common topological surfaces without boundary are spheres and tori.

Cutting and pasting surfaces

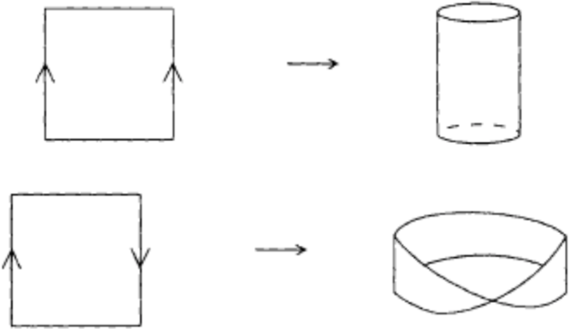
The sphere can be thought of as what you get when you "sew" / "paste" together the boundaries of two disks.



Alternatively, a sphere can be formed by identifying the sides of a two-sided polygon.

Cutting and pasting surfaces

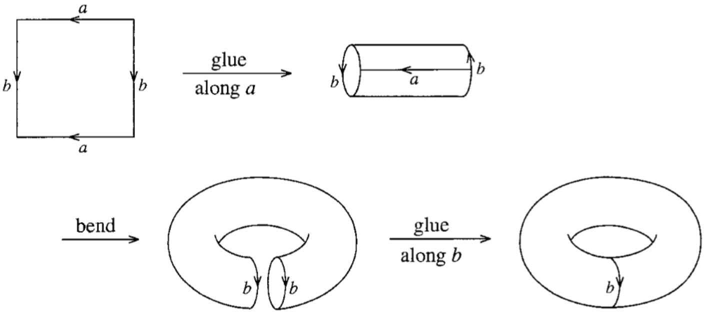
Identifying the sides of a rectangle in different ways gives rise to different surfaces.



Source: [3]

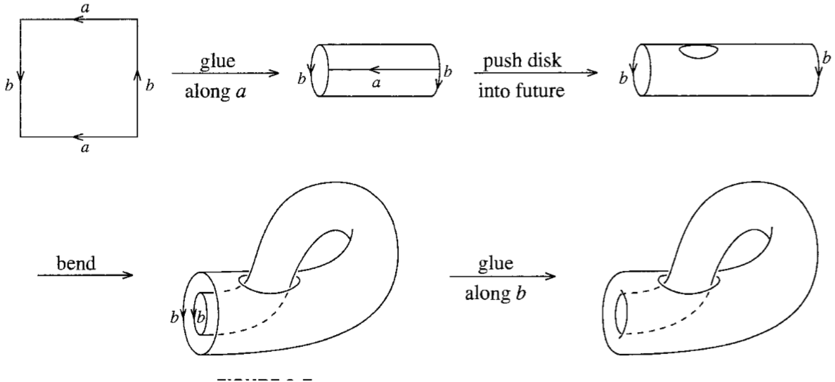
Cutting and pasting surfaces

A torus can be thought of as a rectangle with the sides identified with each other as follows.



Source: [2]

Representing a Klein bottle requires the fourth dimension.



Source: [2]

Representing topological surfaces

It turns out that all topological surfaces without boundary can be represented by identifying pairs of edges in a polygon with an even number of sides.

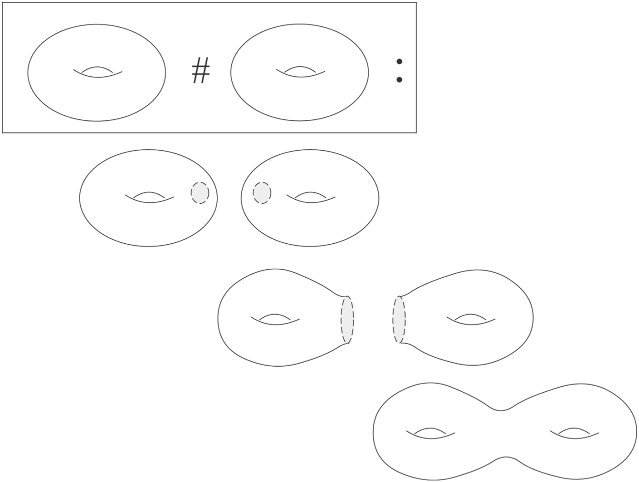
For example, a two-holed torus can be represented as an octagon.

Classification theorem

Any path-connected surface without boundary is homeomorphic to one of the following:

- the sphere
- the connected sum of n tori $(n \geq 1)$
- the connected sum of n projective planes $(n \geq 1)$

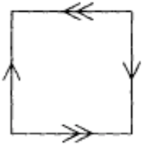
Connected sum of surfaces



Source: [4]

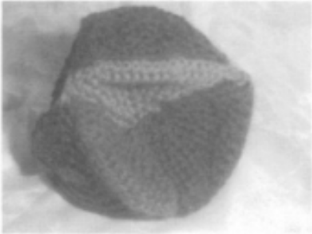
The real projective plane

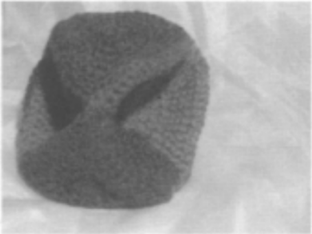
There are multiple ways of representing the real projective plane topologically.

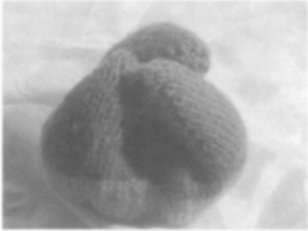


Embeddings and immersions

Claire Irving published knitting and crochet patterns for certain representations of the projective plane in 3-D [3].





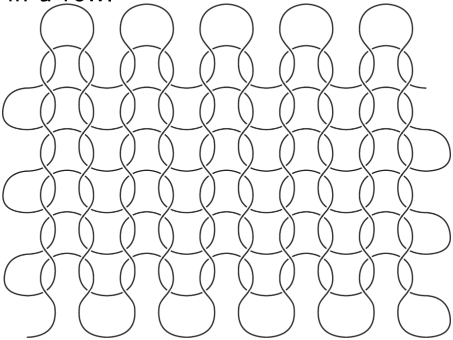


How knitting works

Like crochet, knitted fabric is formed by passing rows of loops through each other.

First, a row of loops is **cast on** to one needle, and the other needle is used to pass loops through these "live" stitches.

Like crochet, knitted fabric can be shaped by increasing or decreasing the number of stitches in a row.



Source: [4]

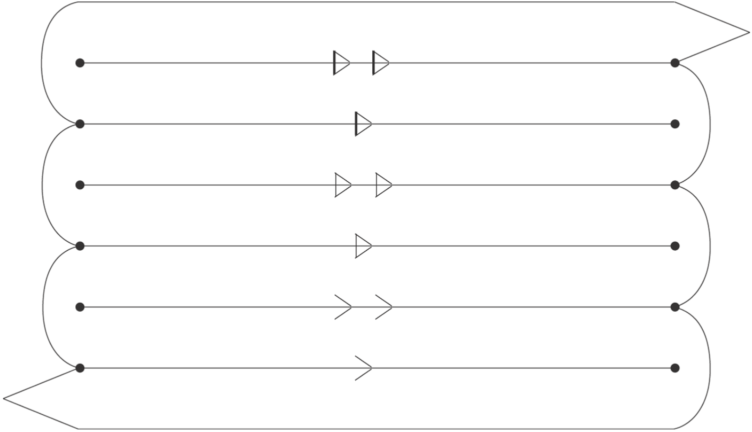
Why knitting?

Knitting offers some advantages in constructing models.

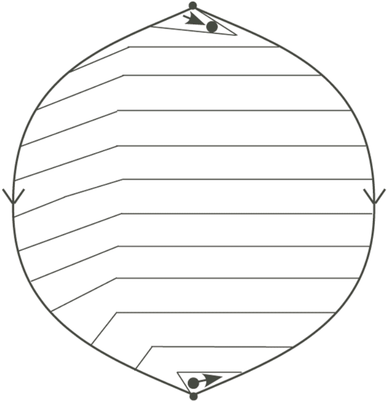
- The cutting and pasting of surfaces can be achieved through **grafting** of knitted fabric.
- It's possible to create surfaces that pass through themselves by passing live stitches through already-knitted fabric in the process of knitting. This is necessary for creating **non-orientable** surfaces without boundary, which must have self-intersection when represented in 3-D space.

Every topological surface can be knit

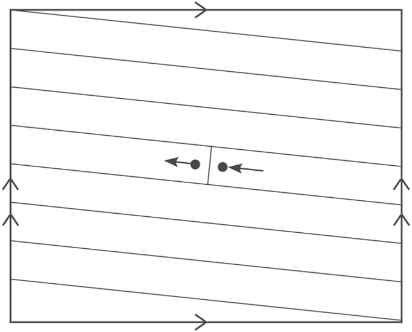
See Sarah-Marie Belcastro's 2009 paper [4] for details.
Belcastro modeled knitting topologically by representing knitted fabric as a long rectangle.



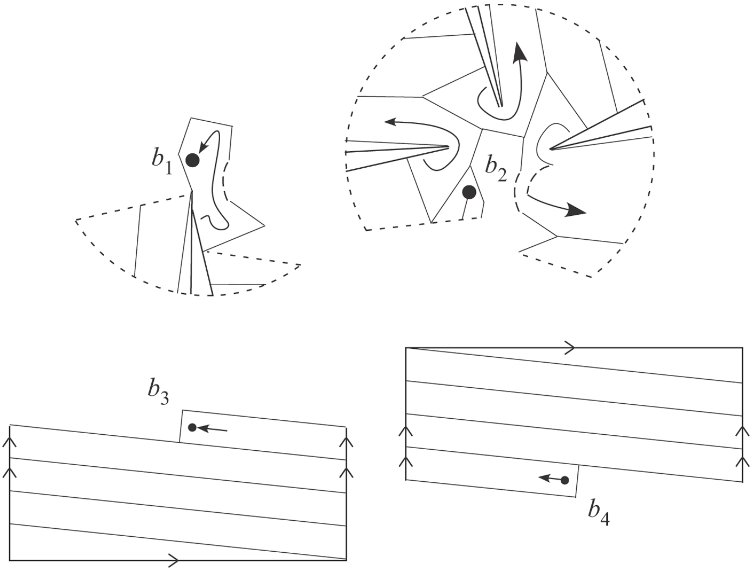
Knitting a sphere



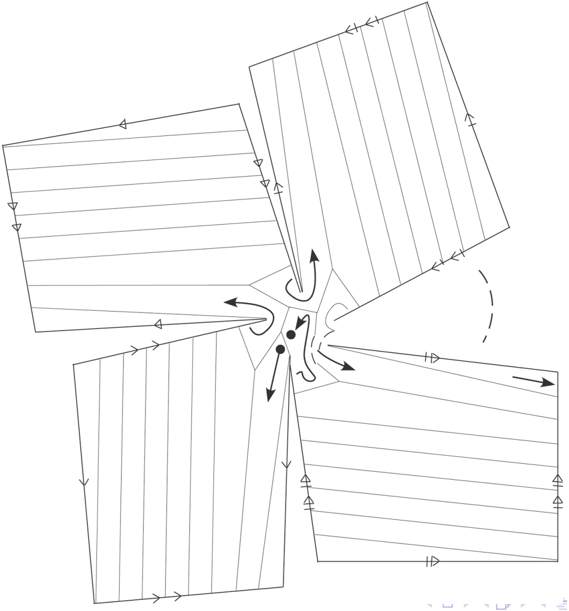
Knitting a torus



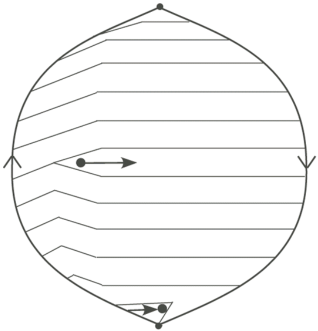
Knitting orientable surfaces



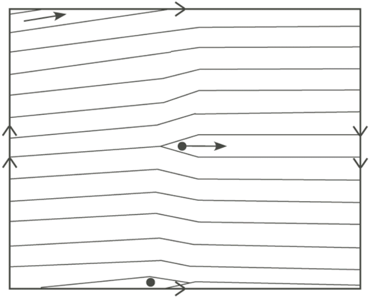
Knitting orientable surfaces



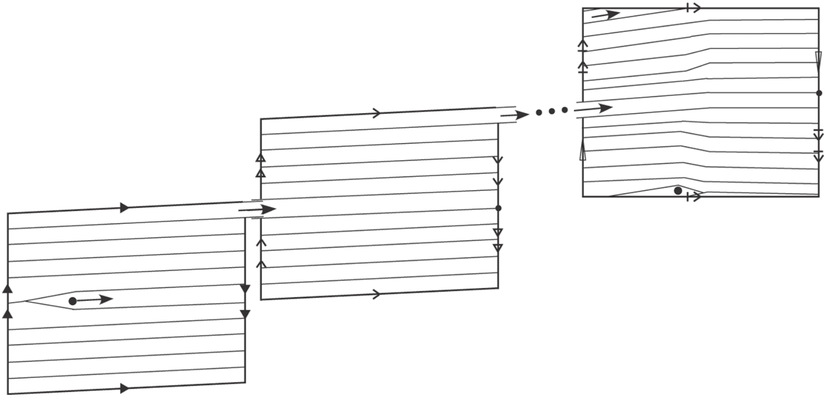
Knitting a projective plane



Knitting a Klein bottle



Knitting non-orientable surfaces



Knitting non-orientable surfaces

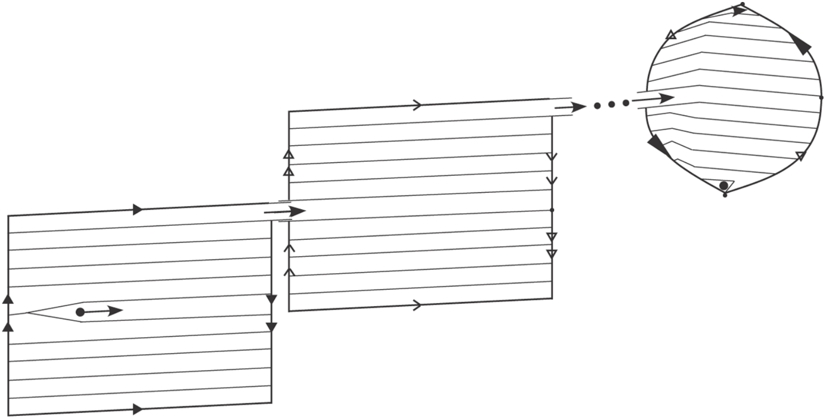


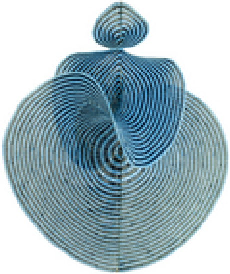
Table of Contents

Seifert surfaces

http://www.toroidalsnark.net/geometricknitting2.pdf

The Lorenz manifold

Hinke Osinga and Bernd Krauskopf used computer-generated instructions to crochet the Lorenz manifold, a representation of the Lorenz equations, which describe chaotic systems.



https://www.math.auckland.ac.nz/~hinke/crochet/

Table of Contents

References

[1] D. Taimina, *Crocheting Adventures with Hyperbolic Planes: Tactile Mathematics, Art and Craft for all to Explore, Second Edition*, 2nd ed. Boca Raton: CRC Press, Aug. 2019.

[2] R. Messer, *Topology now!*, ser. Classroom resource materials. Washington, DC: Mathematical Association of America, 2006.

[3] C. Irving, "Making the Real Projective Plane," *The Mathematical Gazette*, vol. 89, no. 516, pp. 417–423, 2005, publisher: Mathematical Association. [Online]. Available: http://www.jstor.org/stable/3621933

[4] s.-m. belcastro, "Every topological surface can be knit: a proof," *Journal of mathematics and the arts*, vol. 3, no. 2, pp. 67–83, 2009.

Further reading

- On the hyperbolic plane:
  https://theiff.org/oexhibits/oe1.html
- Miles Reid wrote the first work on knitted topological surfaces in 1971: https://homepages.warwick.ac.uk/~masda/knit_surfaces.pdf
- Mark Shoulson's knitted topological surfaces:
  http://web.meson.org/topology/
- The topological zoo: dictionary of 3D computer models of topological objects: http://www.geom.uiuc.edu/zoo/
- Sarah-Marie Belcastro has compiled many references on mathematical knitting and other fiber arts:
  www.toroidalsnark.net/mathknit.html
  www.toroidalsnark.net/mkreflist.html
- There are tons of resources and patterns for beginner knitting and crochet on the internet, such as
  www.knittinghelp.com
  www.reddit.com/r/crochet