Why Geometric Algebra Should be in Standard Linear Algebra Curriculum

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23/03/2022

Our Cast of Characters∗

points, mass	lines, length	plane, area	parallelpiped, volume	… k-volume
combinations, independence	subspaces	basis, span	angles, orthogonality	geometric transformations
scalars	vectors	bivectors	trivectors	k-vectors

The Act of Counting

🍎 🍎 🍎, 🍊 🍊 🍊, 🍊 🍊

$$3\ 🍎 + 3\ 🍊 + 2\ 🍊 = 3\ 🍎 + 5\ 🍊.$$

$$🍎 \times 🍊 = ?$$

What does -1🍎 mean?

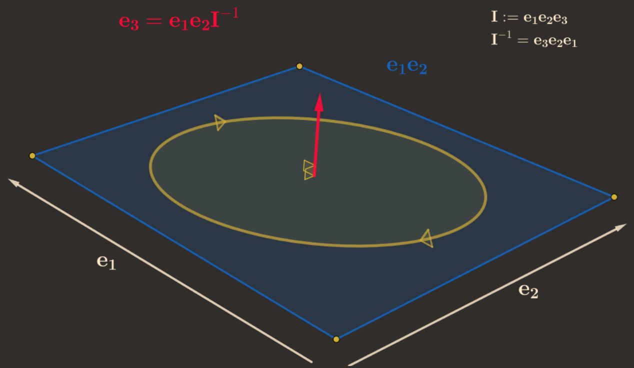
Grassmann's Idea

This may seem contrived, but this is the same principle: $a + bi \in \mathbb{C}$. $a, b \in \mathbb{R}$, or even $\mathbf{v} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + ... + a_n\mathbf{e_n} \in \mathbb{R}^n$.

How far should we go with this? What kind of objects deserve this kind of treatment?

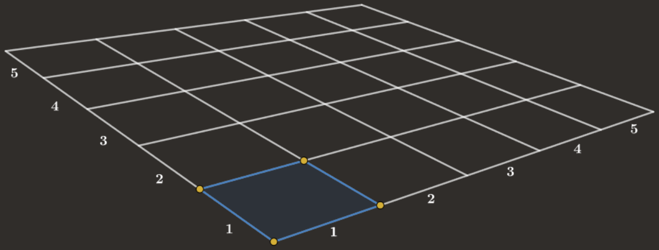
Bivectors $\mathbf{e_1 e_2}$ Represent a Plane*

A *bivector* $\mathbf{B} = \mathbf{u} \wedge \mathbf{v}$ is an *oriented*$(+/-)$ and *shapeless* representation of a plane. It's magnitude $|\mathbf{B}| = |\mathbf{u}||\mathbf{v}| \sin \theta$ is the area of the parallelogram made by the vectors.



Counting Floor Tiles with Vectors and Bivectors

For now, let's just consider $\mathbb{R}^3$.



(Clifford-Grassmann) Geometric Product

$$\vec{\mathbf{u}}\vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \wedge \vec{\mathbf{v}}$$

$$\vec{\mathbf{u}} \wedge \vec{\mathbf{v}} = -\vec{\mathbf{v}} \wedge \vec{\mathbf{u}}$$

In particular $\mathbf{b_1}\mathbf{b_2} = \mathbf{b_1} \wedge \mathbf{b_2}$ if $\mathbf{b_1}, \mathbf{b_2}$ are orthogonal.

Main Idea of Geometric Algebra

Represent *subspaces* of $\mathbb{R}^n$ with algebraic objects in the set $\mathbb{G}^n$.

If the vectors $\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_k}$ are orthogonal, then $\mathbf{b_i} \cdot \mathbf{b_j} = 0$ when $i \neq j$.

$$\implies \mathbf{b_1 b_2 ... b_k} = \mathbf{b_1} \wedge \mathbf{b_2} \wedge ... \wedge \mathbf{b_k}$$

We call these objects *k-blades.* They represent geometrically our arrows, floor tiles, boxes, hyperboxes, etc. as geometric objects of the set $\mathbb{G}^n, k \leq n$.

If we have an element $\mathbf{I}_n \in \mathbb{G}^n$ that is an n-blade, it is called a *pseudoscalar* of $\mathbb{G}^n$, which is unique up to scalar multiplication.

What is $\mathbb{G}^n$?*

$\mathbb{G}^n$ is a 2^n dimensional vector space formed from $\mathbb{R}^n$ by defining a *geometric product* $\mathbf{uv}$ between vectors in $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Basis elements of $\mathbb{G}^3$:

Multivectors*

For the orthonormal basis $\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e_n}$. Let $a_k \in \mathbb{R}$.

$$M = a_0 + \sum_{j=1}^{n} a_j \mathbf{e_j} + \sum_{k=2}^{n} \langle M \rangle_k$$

In $\mathbb{G}^3$, $M = {}^0\vec{s} + {}^1\vec{\mathbf{v}} + {}^2\vec{\mathbf{B}} + {}^3\vec{\mathbf{T}}$.

Result: $\mathbb{G}^n$ is a 2^n dimensional vector space.

Motivation

$$z \in \mathbb{C} \iff z = a + bi,\ i^2 = -1.$$

Something Weird*

We know $\mathbf{e_1 e_2} \in \mathbb{G}^3$.

$$\mathbf{e_i e_j} = \underbrace{\mathbf{e_i} \cdot \mathbf{e_j}}_{0} + \mathbf{e_i} \wedge \mathbf{e_j} = \mathbf{e_i} \wedge \mathbf{e_j} \iff i \neq j.$$

$$\begin{aligned}(\mathbf{e_1 e_2})^2 &= (\mathbf{e_1 e_2})(\mathbf{e_1 e_2}) \\ &= -(\mathbf{e_2}\,\underbrace{\mathbf{e_1 e_1}}_{1}\,\mathbf{e_2}) \\ &= -(\mathbf{e_2 e_2}) \\ &= -1\end{aligned}$$

Complex Numbers $\mathbb{C}$

Let $a, b \in \mathbb{R}$.

$$z = a + b(\mathbf{e_1} \wedge \mathbf{e_2})$$

The set of all $z \in \mathbb{G}^n$ satisfying this statement **is isomorphic to the complex numbers** $(\mathbb{C}, +, \cdot)$.

$\therefore$ The complex numbers are a special case of $\mathbb{G}^n$, and have a better geometric interpretation under this framework.

Why Can $\mathbb{C}$ So Effortlessly Represent Rotations?

Answer (Geometric algebra):

Because there is an element $i \in \mathbb{C}$ that represents the plane which it is rotating on.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Define $\mathbf{i} = \begin{smallmatrix}\text{Unit bivector}\\ \text{containing } \mathbf{u},\mathbf{v}\end{smallmatrix}$.

$$
\begin{aligned}
\mathbf{uv} &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} \\
&= |\mathbf{u}||\mathbf{v}| \cos\theta + |\mathbf{u}||\mathbf{v}| \mathbf{i} \sin\theta \\
&= |\mathbf{u}||\mathbf{v}| (\cos\theta + \mathbf{i} \sin\theta) \\
&= |\mathbf{u}||\mathbf{v}| e^{\mathbf{i}\theta}
\end{aligned}
$$

Which gives the identity $\mathbf{uv} = |\mathbf{u}||\mathbf{v}| e^{\mathbf{i}\theta}$.

In geometric algebra every bivector, and as a result every plane, is a representation of the complex numbers. That means, we can perform rotations easily as in $\mathbb{C}$.

Associating $(^+/-)$ quantities to subspaces of $\mathbb{R}^n$ $+$ Allowing vector multiplication $\iff$ 🤯

Rotating Vectors and Blades in $\mathbb{G}^n$ (*media:paper)

Let $\mathbf{B} \in \mathbb{G}^n$ be a blade. Suppose we want to rotate $\mathbf{B}$ by angle $\theta \in \mathbb{R}$ on a plane represented by the bivector $\mathbf{i}$.

For any blade $\mathbf{B}$ and angle $\mathbf{i}\theta$.

$$\mathsf{R}_{\mathbf{i}\theta}(\mathbf{B}) = e^{-\frac{\mathbf{i}\theta}{2}}\,\mathbf{B}\,e^{\frac{\mathbf{i}\theta}{2}}$$

And vectors in $\mathbf{u} \in \mathbb{R}^3$ are a special case: $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{u}) = e^{-\frac{\mathbf{i}\theta}{2}}\,\mathbf{u}\,e^{\frac{\mathbf{i}\theta}{2}}$ since vectors in $\mathbb{G}^n$ are 1-blades.

Notice that bivectors are our chosen representation of angles, which is by design.

Psuedoscalars

A pseudoscalar of $\mathbb{G}^n$ is an n-blade that represents an orthonormal basis for $\mathbb{R}^n$.

In $\mathbb{G}^2$, $\mathbf{i} = \mathbf{e_1 e_2}$ is a unit pseudoscalar. $(\mathbf{i}^{-1} = -\mathbf{i}.)$

The quaternions $\mathbb{H}$, do not contain a unit pseudoscalar of $\mathbb{G}^3$ because $\mathbf{I_3} = \mathbf{e_1 e_2 e_3}$ cannot be expressed as a product of $\hat{\mathbf{i}} = \mathbf{e_2 e_3}, \hat{\mathbf{j}} = \mathbf{e_1 e_3}, \hat{\mathbf{k}} = \mathbf{e_1 e_2}$.

Why Are They Called Pseudoscalars?

Recall that $\sum_{k=0}^{n} \binom{n}{k} = 2^n = \dim(\mathbb{G}^n)$.

They are called pseudoscalars because they are a basis for a 1-dimensional subspace of $\mathbb{G}^n$ just like the scalar elements of $\mathbb{R}$ are because $\binom{n}{0} = \binom{n}{n} = 1$.

Similarly we have *pseudovectors* which are $(n-2)$*-blades*.

Dual of a Multivector

In $\mathbb{G}^n$, we can obtain the inverse of a unit pseudoscalar $\mathbf{I}_n$ by reversing all of its elements.

$$\implies \mathbf{I}_n^{-1} = \mathbf{I}_n^{\dagger} = (-1)^{\frac{n(n-1)}{2}}\mathbf{I}.$$

For any k-blade $\mathbf{B}$, $\mathbf{B}^* = \mathbf{B}\mathbf{I}_n^{-1}$ is an $(n-k)$-blade that represents the orthogonal complement of the subspace.

For this reason, $(\mathbf{u} \wedge \mathbf{v})^* = \mathbf{u} \times \mathbf{v}$.

But more generally for multivectors $M, N \in \mathbb{G}^n$
$(M \wedge N)^* = M \cdot N^*$, $(M \cdot N)^* = M \wedge N^*$.

Some Housekeeping

$$\mathbf{e}_i^2 = ?$$

Some Housekeeping*

$$\mathbf{e}_i^2 = ?$$

$$\mathbf{e}_i^2 = \begin{cases} 1 \implies \text{Split-complex numbers} \\ -1 \implies \text{Complex numbers} \\ 0 \implies \text{Dual numbers} \end{cases}$$

See [2]: Video: Siggraph2019 Geometric Algebra, to find out what this means in more detail.

For an algebraic reference of Clifford algebras w.r.t. geometric algebra, see the notes [3].

Advantage: Linear Independence

With the inner product we know:

$$\mathbf{u}, \mathbf{v} \text{ are orthogonal} \iff \mathbf{u} \cdot \mathbf{v} = 0.$$

With the outer product:

$$\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n} \text{ are linearly independent} \iff \mathbf{v_1} \wedge \mathbf{v_2} \wedge ... \wedge \mathbf{v_n} \neq 0.$$

Advantage: Orthogonal Complement of a Subspace of $\mathbb{R}^n$ without Matrix Algebra*

Blades represent subspaces of $\mathbb{R}^n$

For any blade $\mathbf{B}$ that represents a subspace $V \subseteq \mathbb{R}^n$. Then $\mathbf{B}^* := \mathbf{B}\mathbf{I}_n^{-1}$ represents the orthogonal complement $V^\perp$ where $\mathbf{I}_n$ is a unit pseudoscalar in $\mathbb{R}^n$.

This is because if $\exists r_j \in \mathbb{R}, \mathbf{u} = \sum_{j=1}^{n} r_j \mathbf{b}_j$ then $\mathbf{u} \wedge \mathbf{b_1} \wedge \mathbf{b_2} \wedge ... \wedge \mathbf{b_n} := 0$, and so if one vector in a wedge product is a linear combination of the others, the whole wedge product goes to 0.

Advantage: Subspace Membership Test With Blades

Let $\mathbf{B} = \mathbf{b}_1 \mathbf{b}_2 \dots \mathbf{b}_n$ be an n-blade representing a subspace $V = \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \dots \mathbf{b}_n\}$ and $\mathbf{u}$ be a 1-vector.

$$\begin{aligned} \mathbf{u} \in V &\iff \mathbf{u} \wedge \mathbf{B} = 0 \\ &\iff \mathbf{u} \cdot \mathbf{B}^* = 0 \\ \mathbf{u} \in V^\perp &\iff \mathbf{u} \wedge \mathbf{B}^* = 0 \\ &\iff (\mathbf{u} \cdot \mathbf{B})^* = 0 \end{aligned}$$

This is the more precise reason that blades are considered to represent subspaces [3](pg. 23, Prop. 3.1) and [1](pg. 122, Thm. 7.2) and that the dual of a blade represents the orthogonal complement of that subspace.

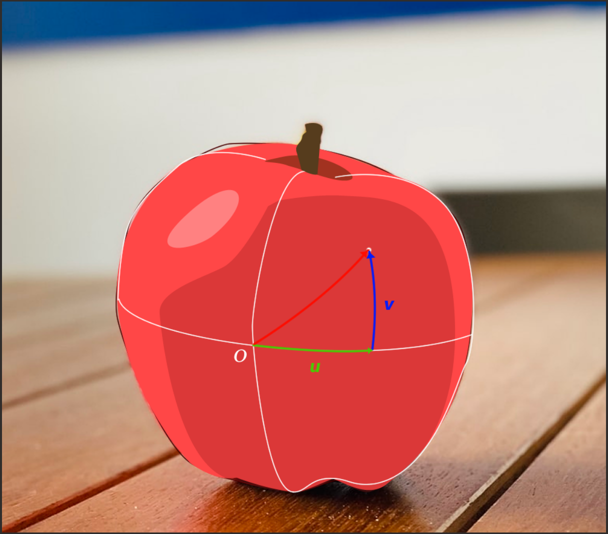
Advantage: Determinants are Fundamental∗

Let $X = \begin{bmatrix} | & | & & | \\ \mathbf{x_1} & \mathbf{x_2} & \dots & \mathbf{x_n} \\ | & | & & | \end{bmatrix}$ where $\mathbf{x_j} \in \mathbb{R}^n$. Then

$$\mathbf{x_1} \wedge \mathbf{x_2} \wedge \dots \wedge \mathbf{x_n} = \det(X)\mathbf{I}_n$$

where $\mathbf{I}_n$ is a unit pseudoscalar in $\mathbb{G}^n$.

Manifolds and Tangent Spaces∗



Advantage: Gradient = Divergence + Curl

For a differentiable field $F : M \to \mathbb{G}^n$ on a manifold M:

$$\nabla F = \underbrace{\nabla \cdot F}_{\text{div}F} + \underbrace{\nabla \wedge F}_{\text{curl}F}$$

Advantage: Multivector Integration (Directed Integrals)*

From [4]. Let $M \subseteq \mathbb{R}^n$ be a nice m-dimensional manifold with parameterization $\mathbf{x}(u_1, u_2, \ldots, u_m) : A \subseteq \mathbb{R}^m \to M \subseteq \mathbb{R}^n$. Let $F : M \to \mathbb{G}^m$.

$$\int_M d^m\mathbf{x}F = \int_A (\mathbf{x}_{u_1} \wedge \mathbf{x}_{u_2} \wedge \cdots \wedge \mathbf{x}_{u_m})F\,dA$$

where $d^m\mathbf{x} = \mathbf{I}_m d^m x$ is the pseudoscalar in the tangent space $T_{\mathbf{p}} \subseteq \mathbb{R}^m$, $\mathbf{p} \in M$ times the infinitesimal m-volume $d^m x$.

Fundamental Theorem of Geometric Calculus

Let M be an m-dimensional manifold (oriented, closed, bounded) with boundary ∂M. For a continuous field $F : M \cup \partial M \to \mathbb{G}^n$. [4].

$$\int_M d^m\mathbf{x}\,\partial F = \oint_{\partial M} d^{m-1}\mathbf{x}\,F.$$

This simple statement also has the following special cases[1] of:
1) Divergence theorem (and so Gauss' Theorem)
2) Curl theorem (and so Green's and Stokes' Theorem)
3) Gradient theorem (and so the FT of line integrals)

[1] For more details see [@macdonald2012vector], Chapter 10, Theorem 10.1, pg. 141-160.

(Bonus) Mathematical Party Tricks∗

Maxwell's equations becomes *Maxwell's equation*.

From [4] pg. 66:

$$\begin{cases} \nabla \cdot \mathbf{e} = 0 \\ \nabla \wedge \mathbf{B} = 0 \\ \nabla \wedge \mathbf{e} = -\partial_t \mathbf{B} \\ \nabla \cdot \mathbf{B} = -\partial_t \mathbf{e} \end{cases} \iff \begin{matrix} (\partial_t + \nabla)F = 0 \\ F := \mathbf{e} + \mathbf{B} \end{matrix}$$

The ∂_t is the component of the gradient ∇ of the variable t.

Why Have I Never Heard of This?

Long story short: For historical reasons, Clifford's work did not become as well known among mathematicians as people like Gibbs.

Video: The Vector Algebra War

Paper: The Vector Algebra War: A Historical Perspective

→ "We thus historically review the development of our various vector systems and conclude that Clifford's multivectors best fulfills the goal of describing vectorial quantities in three dimensions and providing a unified vector system for science." [5]

From $\mathbb{R}^n$ to $\mathbb{G}^n$*

To get from $\mathbb{R}^n$ to $\mathbb{G}^n$ you only have to accept the following:

- Closure under the geometric product AB, $A, B \in \mathbb{G}^n$.
  (Associative, distributive, homogeneous, with unity $1 \in \mathbb{R}$)

For a short and barebones elementary construction of $\mathbb{G}^n$, see [6].

When Should Geometric Algebra be Taught?

My belief: after the dot product in $\mathbb{R}^n$ and before the discussion of *planes* or systems of equations.

This $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$ is simple enough for first year students and leads to a generalizable framework for complex numbers and quaternions.

Most basic idea: Represent subspaces of $\mathbb{R}^n$ algebraically with *blades*, which are products of orthogonal vectors.

The Most Important Calculation

Let $\mathbf{u} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{v} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. $a_i, b_i \in \mathbb{R}$.

Show as exercise:

$$
\begin{aligned}
\mathbf{uv} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\
&= (a_1b_1 + a_2b_2 + a_3b_3) \\
&+ (a_2b_3 - a_3b_2)\mathbf{e}_2\mathbf{e}_3 \\
&+ (a_3b_1 - a_1b_3)\mathbf{e}_3\mathbf{e}_1 \\
&+ (a_1b_2 - a_2b_1)\mathbf{e}_1\mathbf{e}_2 \\
&= \mathbf{u} \cdot \mathbf{v} + (\mathbf{u} \times \mathbf{v})\,\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1
\end{aligned}
$$

Is it Worth it?

Great visual introduction in `manim`:

Video: A Swift Introduction to Geometric Algebra by *sudgylacmoe*.

For a more historical perspective, see:

Video: David Hestenes - Tutorial on Geometric Calculus [7].

Recommended Textbooks for Undergraduates?

[1] Amazon.ca: Linear and Geometric Algebra, Alan Macdonald
Video Playlist Linear and Geometric Algebra, Alan Macdonald

[4] Amazon.ca: Vector and Geometric Calculus, Alan Macdonald
Video Playlist Geometric Calculus, Alan Macdonald

Why I like them: Short, cheap, concise, filled to the brim with exercises. Associated videos from the author. Fantastic cost/value.

Extras

- bivector.net: Awesome website with lots of resources and web animations made using geometric algebra.
- Colour Palette: #302D2A, #D4AF37, #FF0A60, #156581, #FFEACB.

References

[1] A. Macdonald, *Linear and geometric algebra*. Alan Macdonald, 2010.

[2] Bivector, *Siggraph2019 geometric algebra*, (2019).Available: https://www.youtube.com/watch?v=tX4H_ctggYo

[3] D. Lundholm and L. Svensson, "Clifford algebra, geometric algebra, and applications," *arXiv preprint arXiv:0907.5356*, 2009.

[4] A. Macdonald, *Vector and geometric calculus*, vol. 12. CreateSpace Independent Publishing Platform, 2012.

[5] J. M. Chappell, A. Iqbal, J. G. Hartnett, and D. Abbott, "The vector algebra war: A historical perspective," *IEEE Access*, vol. 4, pp. 1997–2004, 2016, doi: 10.1109/ACCESS.2016.2538262.

[6] A. Macdonald, "An elementary construction of the geometric algebra," *Advances in applied Clifford algebras*, vol. 12, no. 1, pp. 1–6, 2002.

[7] N. Nominandum, *David hestenes - tutorial on geometric calculus*, (2015).Available: https://www.youtube.com/watch?v=ItGlUbFBFfc

[8] D. Hestenes and R. Ziegler, "Projective geometry with clifford algebra," *Acta Applicandae Mathematica*, vol. 23, no. 1, pp. 25–63, 1991.