Influence Centrality

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Introduction

Notion of Centrality

Centrality refers to a measure of how important or well-positioned a node is in a network.

Different Types of Centrality

- Degree Centrality
- Closeness Centrality
- Betweenness Centrality
- Eigenvector Centrality
- Katz Centrality
- PageRank Centrality

Closeness Centrality

let $d(v_i, v_j)$ denote the distance (length of shortest path) from v_i to v_j;

$$C_C(v_i) = \frac{1}{\sum_{j=1, j \neq i}^{n} d(v_i, v_j)}$$

Closeness Centrality

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$$C_C(v_i) = \frac{1}{\sum_{j=1, j\neq i}^{n} d(v_i, v_j)}$$

$$C'_C(v_i) = \frac{n-1}{\sum_{j=1, j\neq i}^{n} d(v_i, v_j)}$$

Betweenness Centrality

Let $g_{j,k}$ represent the number of geodesics (shortest paths) between v_j and v_k. And let $g_{j,k}(v_i)$ be the number of geodesics between v_j and v_k that pass through v_i.

$$C_B(v_i) = \sum_{j<k}^{n} \frac{g_{j,k}(v_i)}{g_{j,k}}$$

Betweenness Centrality

Let $g_{j,k}$ represent the number of geodesics (shortest paths) between v_j and v_k. And let $g_{j,k}(v_i)$ be the number of geodesics between v_j and v_k that pass through v_i.

$$C_B(v_i) = \sum_{j<k}^{n} \frac{g_{j,k}(v_i)}{g_{j,k}}$$

$$C'_B(v_i) = \frac{2}{(n-1)(n-2)} \sum_{j<k}^{n} \frac{g_{j,k}(v_i)}{g_{j,k}}$$

Influence Centrality

Let G be a simple weighted directed graph with n nodes.

Suppose that all arcs in G are (or can be standardized).

suppose further the weighted indegree sum of any node v_i is less than or equal to 1; i.e. $\sum_{j=1}^{n} w(e_{ji}) \leq 1$ for any node v_i.
where e_{ji} represents an arc from vertex v_j to vertex v_i. $w(e_{ji})$ denotes the weight of the edge e_{ji}.

Lastly, the nodes have to be assigned standardized values that represent the relative value of the resources directly under a node's influence/access (this is based on information that is not from the graph itself). Denote the value of node v_i by $V(v_i)$.

Influence Centrality

Power (total influence)

$$P(v_i) = V(v_i) + \sum_{j=1}^{n} w(e_{ij}) \cdot P(v_j) \tag{1}$$

Influence Centrality

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Internal Influence

$$I(v_i) = 1 - \sum_{k=1}^{n} w(e_{ki})$$

Influence Centrality

Power (total influence)

$$P(v_i) = V(v_i) + \sum_{j=1}^{n} w(e_{ij}) \cdot P(v_j) \qquad (1)$$

Internal Influence

$$I(v_i) = 1 - \sum_{k=1}^{n} w(e_{ki})$$

Influence Centrality

$$C_I(v_i) = I(v_i) \cdot P(v_i) \qquad (2)$$

Power Equations

Power Equations

Recall equation (1)

$$P(v_i) = V(v_i) + \sum_{j=1}^{n} w(e_{ij}) \cdot P(v_j)$$

Note that this is a system of n equations with n unknowns.

Power Equations: Matrix Form

let $A = \{a_{ij}\}$ be the adjacency matrix of the graph G, and let $V = \{V(v_i)\}$ be the vector of values for the nodes, and $P = \{P(v_i)\}$ be the vector of powers for the nodes;

$$\mathbf{V} = \begin{pmatrix} V(v_1) \\ V(v_2) \\ \vdots \\ V(v_n) \end{pmatrix}, \mathbf{P} = \begin{pmatrix} P(v_1) \\ P(V_2) \\ \vdots \\ P(v_n) \end{pmatrix}$$

and,

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} = \begin{pmatrix} w(e_{1,1}) & w(e_{1,2}) & \dots & w(e_{1,n}) \\ w(e_{2,1}) & w(e_{2,2}) & \dots & w(e2,n) \\ \vdots & \vdots & \ddots & \vdots \\ w(e_{n,1}) & w(e_{n,2}) & \dots & w(e_{n,n}) \end{pmatrix}$$

Power Equations: Matrix Form

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} = \begin{pmatrix} w(e_{1,1}) & w(e_{1,2}) & \dots & w(e_{1,n}) \\ w(e_{2,1}) & w(e_{2,2}) & \dots & w(e2,n) \\ \vdots & \vdots & \ddots & \vdots \\ w(e_{n,1}) & w(e_{n,2}) & \dots & w(e_{n,n}) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & w(e_{1,2}) & \dots & w(e_{1,n}) \\ w(e_{2,1}) & 0 & \dots & w(e2,n) \\ \vdots & \vdots & \ddots & \vdots \\ w(e_{n,1}) & w(e_{n,2}) & \dots & 0) \end{pmatrix}$$

Power Equations: Matrix Form

then, the system we want to solve is given by:

$$(\mathbf{A} - \mathbf{I}_n) \cdot \mathbf{P} = -\mathbf{V}$$

Or,

$$(\mathbf{I}_n - \mathbf{A}) \cdot \mathbf{P} = \mathbf{V} \tag{3}$$

Power Equations: Walks Method

Power Equation

Recall equation (1)

$$P(v_i) = V(v_i) + \sum_{j=1}^{n} w(e_{ij}) \cdot P(v_j)$$

Suppose we expand all the P(v) terms in the RHS, to obtain the LHS in terms of values and weights of arcs.

Power Equations: Walks Method

Let W_{ij} denote the set of all walks from v_i to v_j, and let ${}_mW_{ij}$ denote the m^{th} element, for m = 1, 2, ..., n.

$$P(v_i) = V(v_i) + (\sum_{j=1}^{n}(\sum_{m=1}^{n_{ij}}(\prod_{e \in {}_mW_{ij}} w(e))) \cdot V(v_j)) \tag{4}$$

Influence Centrality Properties

Property 1

$C_I(v_i) \geq 0, \quad \forall i = 1, 2, ..., n.$

Property 2

$\sum_{i=1}^{n} C_I(v_i) = \sum_{i=1}^{n} V(v_i)$. That is, the sum of the influence centrality of all nodes in a network, G, is equal to the sum of the values of all nodes in G.

Property 3

$C_I(v_i) \leq \sum_{i=1}^{n} V(v_i), \ \forall\, i = 1, 2, ..., n.$

Some Additional Vectors

the internal influence vector, $\mathcal{I} = \{I(v_i)\}_{i=1}^{n}$, and the $1 \times n$ vector of 1s, $\mathbf{D}$

$$\mathcal{I} = \begin{pmatrix} I(v_1) \\ I(v_2) \\ \vdots \\ I(v_n) \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1, & 1, & \ldots, & 1 \end{pmatrix}$$

Some Additional Vectors

Recall how internal influence, I(v), is defined:

$$I(v_i) = 1 - \sum_{j=1}^{n} w(e_j i) = 1 - \sum_{j=1}^{n} a_{ji}$$

This can be written in matrix form as:

$$\mathcal{I}^T = \mathbf{D} - \mathbf{DA} = \mathbf{D}(\mathbf{I}_n - \mathbf{A}) \tag{5}$$

Standardizing Influence Centrality

standardizing over n gives a measure of how much influence a node v_i has from, or in, their network, taking into account how rich or valuable the network itself is.
so, we define both, for use in different cases:
Standardizing influence centrality for a node v_i in a graph with n nodes (both notions):

$$C_I'(v_i) = \frac{I(v_i) \cdot P(v_i)}{\sum_{m=1}^{n} V(v_m)}$$

$$C_I''(v_i) = \frac{I(v_i) \cdot P(v_i)}{n}$$

Centralization

Measuring Centralization in a graph G with n nodes (both notions):

$$C_I(G) = \frac{\sum_{i=1}^{n} C_I(v_{i^*}) - C_I(v_i)}{(n-1) \cdot \sum_{m=1}^{n} V(v_m)}$$

$$C_I'(G) = \frac{\sum_{i=1}^{n} C_I(v_{i^*}) - C_I(v_i)}{n \cdot (n-1)}$$

Extension: Weighted Measures

Suppose we are comparing two or more networks, or possible structures for the same network.
Suppose we have a function, p, that maps every node to a non-negative real number, preferably in the interval [0, 1] (standardize when possible), so that p is a measure of some property the nodes have.
then we can use a corresponding weighted measure, p', for the nodes, by taking:

$$p'(v) = p(v) * C_I'(v)$$

where C_I' could be either one of the two notions for standardizing centrality. Then, we can take the weighted average with respect to centrality as a measure of the degree to which the graph represents this property (with standardization over values, the denominator is 1, due to property 2)

Reaction Speed and Stability

Take the reaction speed, $r(v)$ to be:

$$r(v_i) = I(v_i) = 1 - \sum j = 1^n w(e_{ij})$$

$$r'(v_i) = I(v_i) \cdot C_I'(v_i)$$

and the Stability of a node, S(v), defined as:

$$s(v_i) = 1 - \frac{\sum_{j=1}^{n} w(e_{ij})}{n}$$

$$s'(v_i) = (1 - \frac{\sum_{j=1}^{n} w(e_{ij})}{n}) \cdot C_I'(v_i)$$

Generalization

Influence centrality can be extended to multi-digraphs with multiple directed relations representing different things, one would simply measure influence centrality and centralization separately for each relation. However, note that since the nodes are valued, and the value is relevant to the relation, it would usually make more sense to use multiple graphs with the same nodes (but possibly different values), each with a single relation.

We can also take the average centrality of each node over the multiple relations.

For example, 'sells to' can be paired with 'buys from' (similarly, exports with imports), or social influence can be paired with formal influence.

Examples

Some example graphs and tables are at the bottom

Example Applications

- e_{ij} = influence over, $V(v)$ = v's direct resources
- e_{ij} = buys from, $V(v)$ = v's total production
- e_{ij} = sells to, $V(v)$ = v's total spending
- e_{ij} = traffic, $V(v)$ = population / crowd

Eigenvector Centrality and Variations

$$C_E(v_i) = \kappa \cdot \sum_{j=1}^{n} a_{j,i} \cdot C_E(v_i)$$

$$C_k(v_i) = \beta + \alpha \cdot \sum_{j=1}^{n} a_{j,i} \cdot C_K(v_i)$$

$$C_P(v_i) = \beta + \alpha \cdot \sum_{j=1}^{n} \frac{a_{j,i}}{d(v_j)} \cdot C_P(v_i)$$

$$C_P(v_i) = \alpha \cdot \sum_{j=1}^{n} \frac{a_{j,i}}{d(v_j)} \cdot C_P(v_i)$$

where $d_o(v_i)$ denotes the outdegree of v_i (not weighted). And $a_{j,i}$ still refers to the $(j, i)^{th}$ element of the adjacency matrix $\mathbf{A}$.

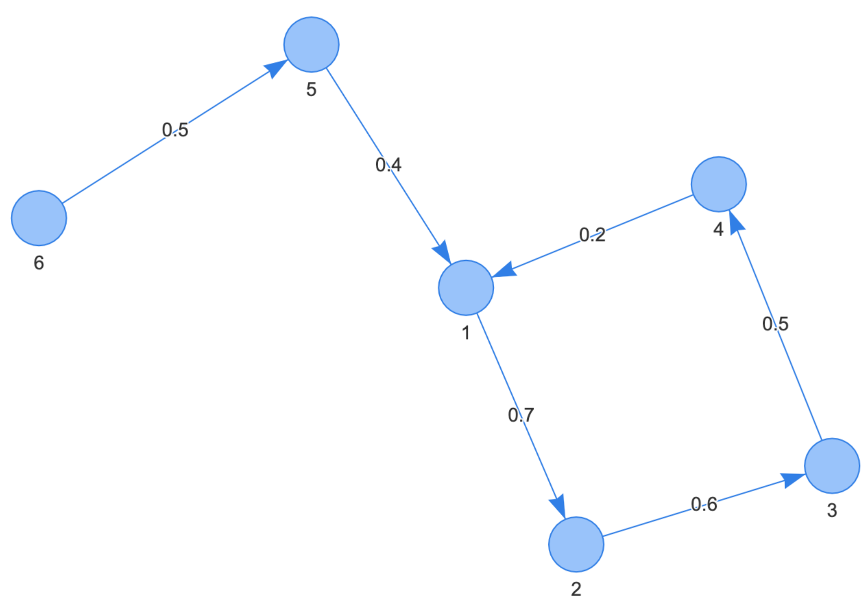
References

Stanley Wasserman and Katherine Faust
Social Network Analysis Methods and Applications
p.152-153, 169-215

Mark Newman
Networks, second edition
p. 110, 159-167

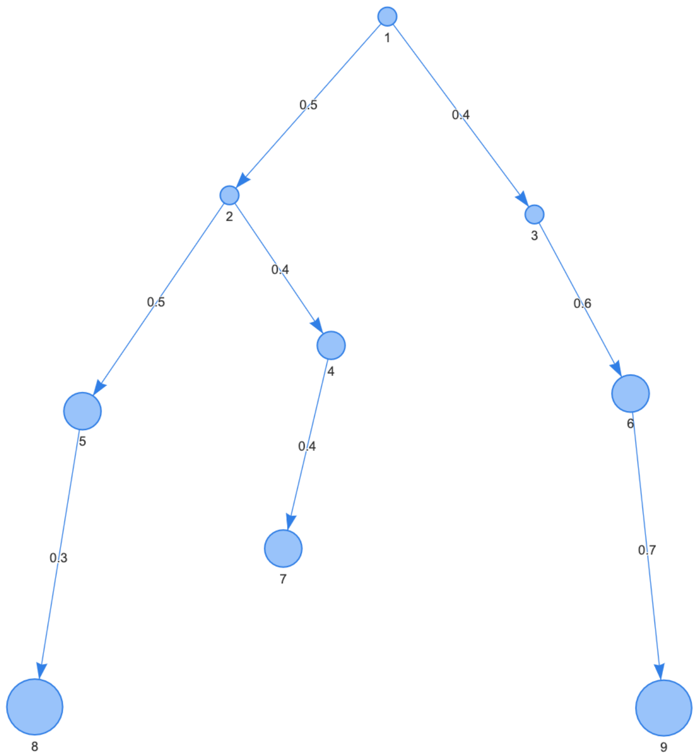
Github repository for program to compute influence centrality in a network.
https://github.com/rakan-omar/Influence-Centrality

Examples



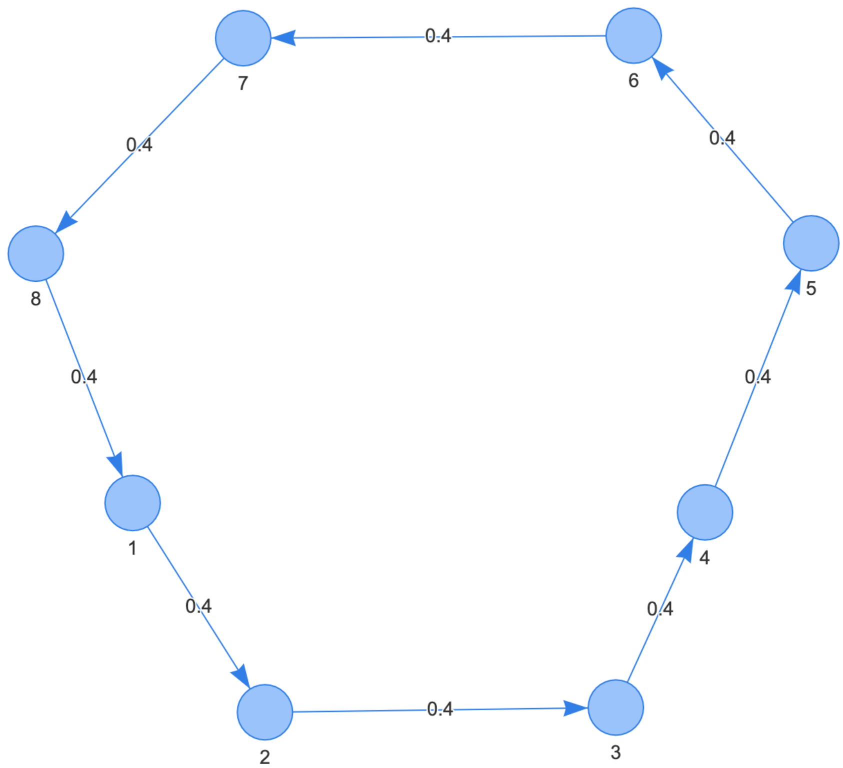
Graph 1

node (v_i)	$V(v_i)$	$C_I(v_i)$	$C'_I(v_i)$	$C''_I(v_i)$	$r'(v_i)$	$s'(v_i)$
1	1	0.97286	0.16214	0.16214	0.06486	0.14323
2	1	0.61378	0.1023	0.1023	0.03069	0.09207
3	1	0.69729	0.11621	0.11621	0.04649	0.10653
4	1	0.74322	0.12387	0.12387	0.06193	0.11974
5	1	0.98643	0.16441	0.16441	0.0822	0.15344
6	1	1.98643	0.33107	0.33107	0.33107	0.30348
G	6	6.0	0.19729	0.19729	0.61724	0.91849



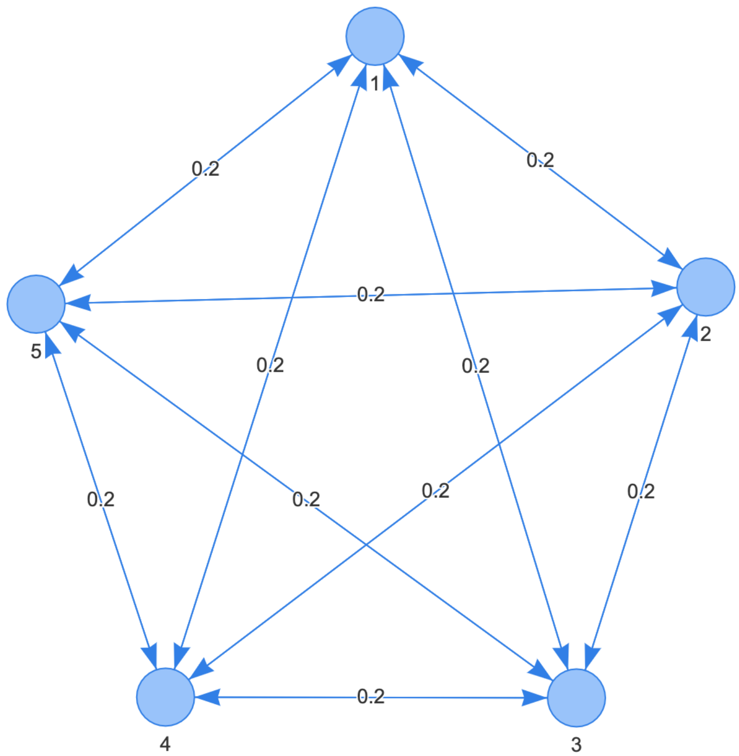
Graph 2

node (v_i)	$V(v_i)$	$C_I(v_i)$	$C'_I(v_i)$	$C''_I(v_i)$	$r'(v_i)$	$s'(v_i)$
1	0.5	1.6877	0.28128	0.18752	0.28128	0.25315
2	0.5	0.6685	0.11142	0.07428	0.05571	0.10027
3	0.5	0.7788	0.1298	0.08653	0.07788	0.12115
4	0.6	0.528	0.088	0.05867	0.0528	0.08409
5	0.7	0.485	0.08083	0.05389	0.04042	0.07814
6	0.7	0.532	0.08867	0.05911	0.03547	0.08177
7	0.7	0.42	0.07	0.04667	0.042	0.07
8	0.9	0.63	0.105	0.07	0.0735	0.105
9	0.9	0.27	0.045	0.03	0.0135	0.045
G	6.0	6.0	0.19144	0.12763	0.67255	0.93857



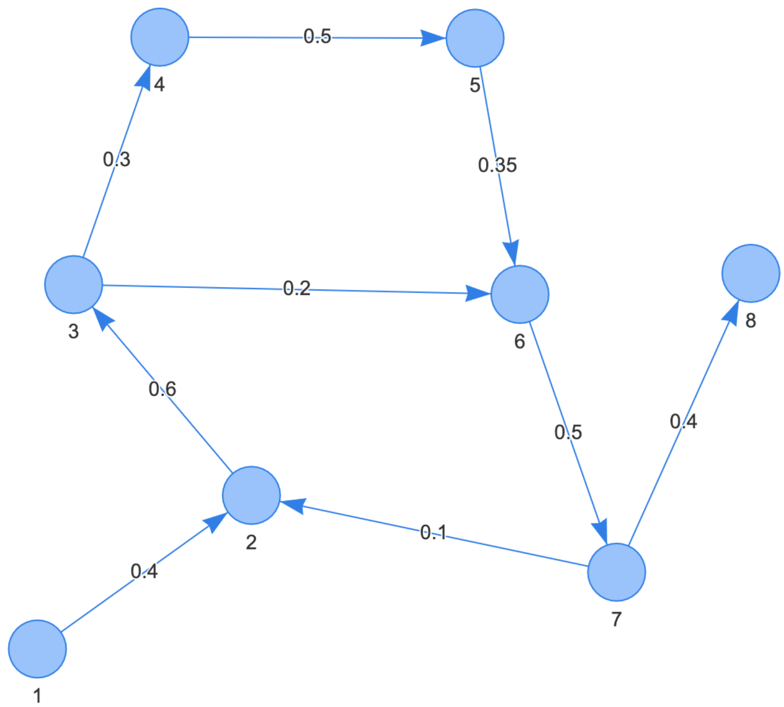
Graph 3

node (v_i)	$V(v_i)$	$C_I(v_i)$	$C'_I(v_i)$	$C''_I(v_i)$	$r'(v_i)$	$s'(v_i)$
1	1	1.0	0.125	0.125	0.075	0.11875
2	1	1.0	0.125	0.125	0.075	0.11875
3	1	1.0	0.125	0.125	0.075	0.11875
4	1	1.0	0.125	0.125	0.075	0.11875
5	1	1.0	0.125	0.125	0.075	0.11875
6	1	1.0	0.125	0.125	0.075	0.11875
7	1	1.0	0.125	0.125	0.075	0.11875
8	1	1.0	0.125	0.125	0.075	0.11875
G	8	8.0	0.0	0.0	0.6	0.95



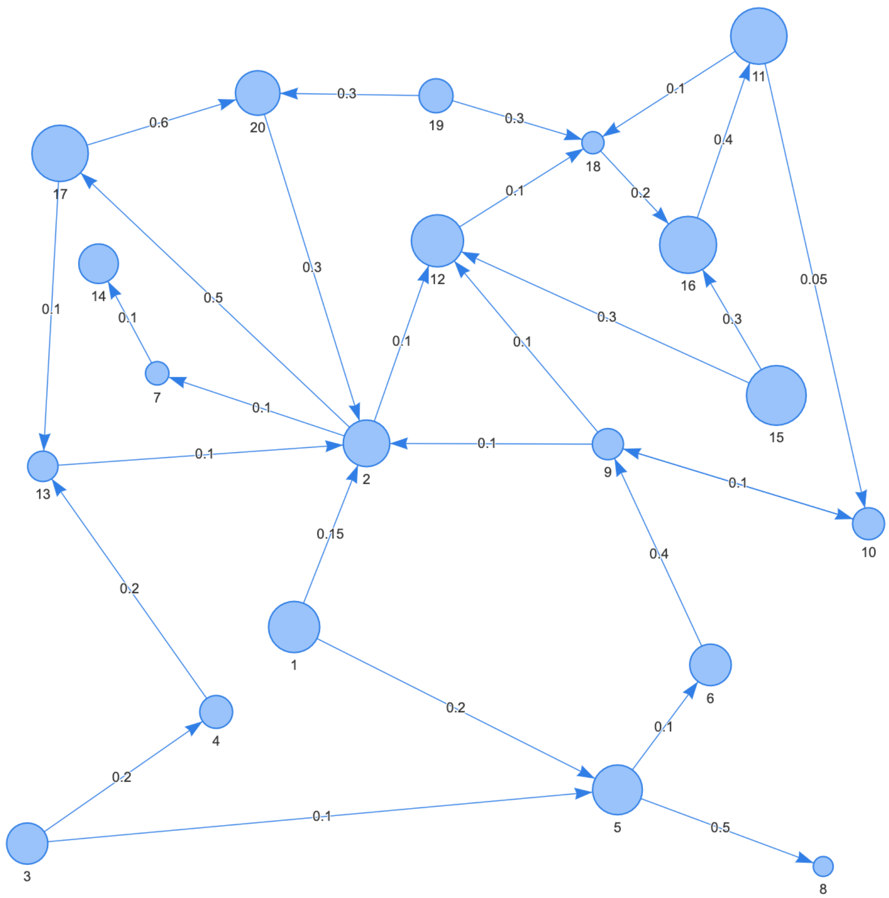
Graph 4

node (v_i)	$V(v_i)$	$C_I(v_i)$	$C'_I(v_i)$	$C''_I(v_i)$	$r'(v_i)$	$s'(v_i)$
1	1	1.0	0.2	0.2	0.04	0.168
2	1	1.0	0.2	0.2	0.04	0.168
3	1	1.0	0.2	0.2	0.04	0.168
4	1	1.0	0.2	0.2	0.04	0.168
5	1	1.0	0.2	0.2	0.04	0.168
G	5	5.0	0.0	0.0	0.2	0.84



Graph 5

node (v_i)	$V(v_i)$	$C_I(v_i)$	$C'_I(v_i)$	$C''_I(v_i)$	$r'(v_i)$	$s'(v_i)$
1	1	1.85752	0.23219	0.23219	0.23219	0.22058
2	1	1.07189	0.13399	0.13399	0.06699	0.12394
3	1	0.76253	0.09532	0.09532	0.03813	0.08936
4	1	1.27138	0.15892	0.15892	0.11125	0.14899
5	1	0.81626	0.10203	0.10203	0.05102	0.09757
6	1	0.81324	0.10165	0.10165	0.04574	0.0953
7	1	0.80719	0.1009	0.1009	0.05045	0.09459
8	1	0.6	0.075	0.075	0.045	0.075
G	8	8.0	0.1225	0.1225	0.64076	0.94533



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Graph 6

node (v_i)	$V(v_i)$	$C_I(v_i)$	$C'_I(v_i)$	$C''_I(v_i)$	$r'(v_i)$	$s'(v_i)$
1	0.77578	1.17485	0.11079	0.05874	0.11079	0.10885
2	0.66525	0.54613	0.0515	0.02731	0.01802	0.0497
3	0.53373	0.6994	0.06595	0.03497	0.06595	0.06496
4	0.33057	0.33264	0.03137	0.01663	0.02509	0.03105
5	0.74204	0.57756	0.05446	0.02888	0.03812	0.05283
6	0.54042	0.68946	0.06502	0.03447	0.05851	0.06372
7	0.0996	0.13401	0.01264	0.0067	0.01137	0.01257
8	0.01289	0.00644	0.00061	0.00032	0.0003	0.00061
9	0.28837	0.28206	0.0266	0.0141	0.0133	0.0262
10	0.30837	0.31007	0.02924	0.0155	0.02485	0.02909
11	0.90108	0.57131	0.05387	0.02857	0.03232	0.05347
12	0.79958	0.41623	0.03925	0.02081	0.01962	0.03905
13	0.27014	0.29832	0.02813	0.01492	0.01969	0.02799
14	0.49298	0.44368	0.04184	0.02218	0.03766	0.04184
15	0.983	1.622	0.15295	0.0811	0.15295	0.14836
16	0.91668	0.64878	0.06118	0.03244	0.03059	0.05996
17	0.90166	0.79697	0.07515	0.03985	0.03758	0.07252
18	0.0692	0.16435	0.0155	0.00822	0.00775	0.01534
19	0.35859	0.78204	0.07375	0.0391	0.07375	0.07153
20	0.61467	0.10828	0.01021	0.00541	0.00102	0.01006
G	10.6046	10.6046	0.10837	0.05746	0.77926	0.97971