Making Juggling Mathematical

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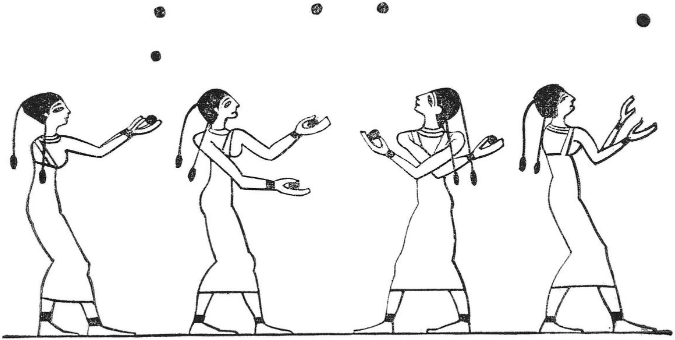
Juggling Is Old!

Oldest known depictions appear in an Egyptian temple at Beni Hasan (c. 1994-1781 BCE).



A Numerical Description for Juggling Patterns

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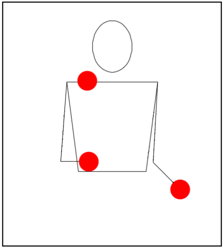
- Only in the 1980s did jugglers develop a way to keep track of different juggling patterns mathematically.
- Idea: use a numerical code to describe the throws.
- Measure height of throw according to number of "beats" until it comes back down (usually, "beats" = "thuds")



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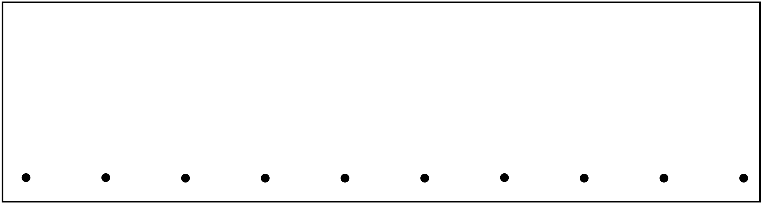

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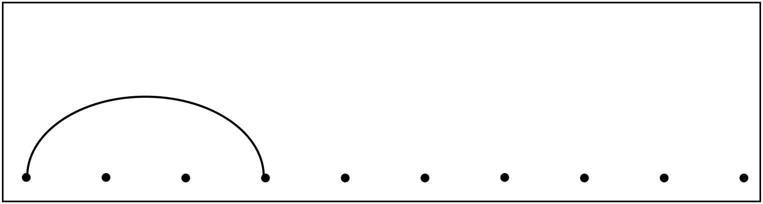

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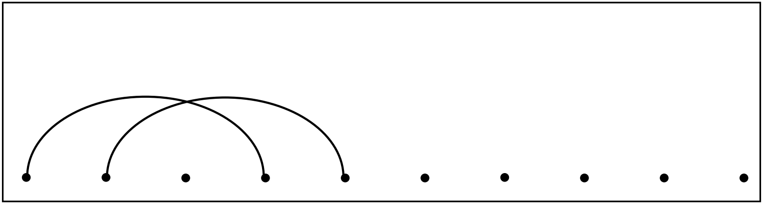

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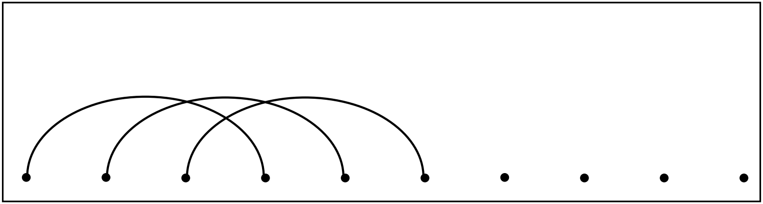

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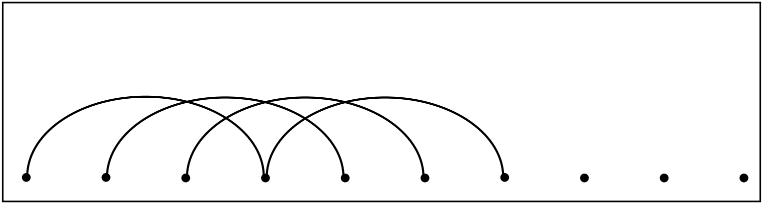

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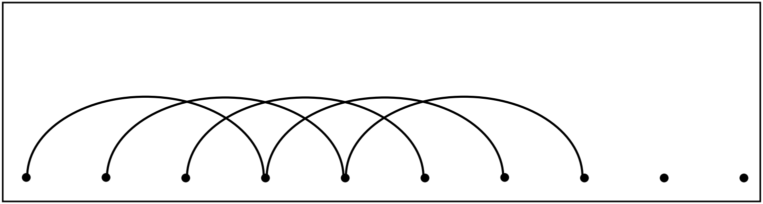

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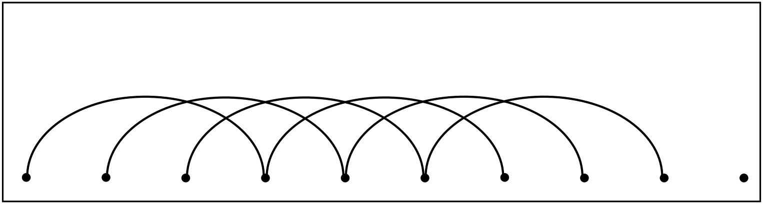

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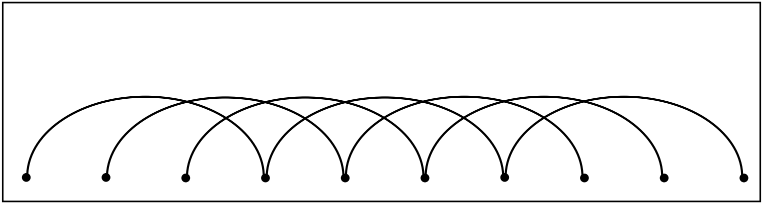

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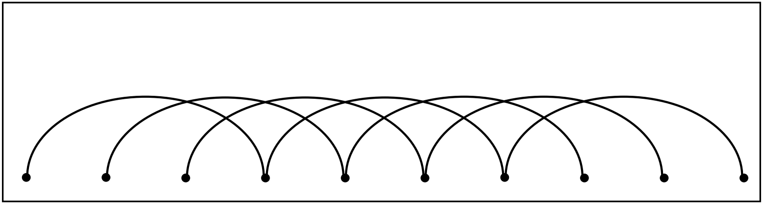

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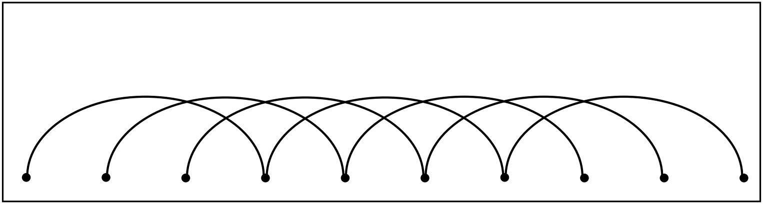
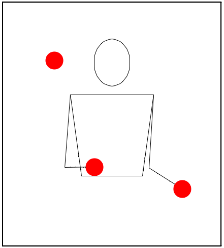
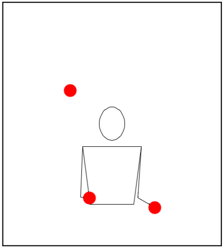


- Repeated throws of height 3.
- This pattern can be represented by $(\ldots 3333 \ldots)$, or just (3).
- This is the *siteswap* for the juggling pattern.

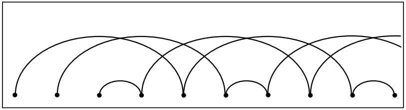
Introduction to Siteswap Notation

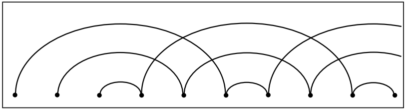


(441)



(531)





Properly Defining A Siteswap

Some remarks:

- The beats always alternate between left and right hands.
- The *length* (or, *period*) of a siteswap is the number of beats that occur before it repeats.
- We are only interested in *monoplex* juggling: at most one ball caught/thrown at once.
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More Precisely: If two balls are thrown at times i and j, and remain in the air for t_i beats and t_j beats, respectively, it cannot be the case that $t_i + i = t_j + j$ (since this would create a collision).

Counting Siteswaps

Definition

A *siteswap* is a finite sequence of nonnegative integers. A *valid* siteswap is one with no collisions, i.e., the quantities $t_i + i \pmod{n}$ are distinct for $1 \leq i \leq n$.

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$$(531) \to \frac{5+3+1}{3} = 3 \text{ balls}, \quad (51635) \to \frac{5+1+6+3+5}{5} = \frac{20}{5} = 4 \text{ balls}.$$

The Reverse Question

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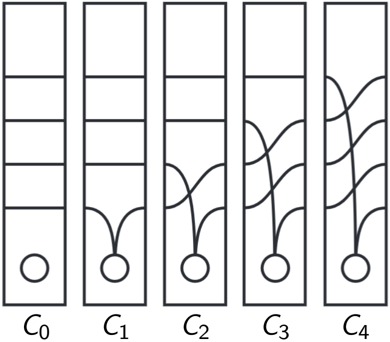
Examples:

- For $b = 2$ and $n = 2$, there are five: $(22), (40), (04), (31), (13)$.
- For $b = 3$ and $n = 3$, there are 37:

$$(900), (090), (009), (630), (603), (063), (360), (036), (306), (333), (711), (171), (117), (441), (414), (144), (522), (252), (225), (720), (180), (126), (450), (423), (153), (027), (018), (612), (045), (342), (351), (702), (801), (261), (504), (234), (135)!$$

Juggling Cards

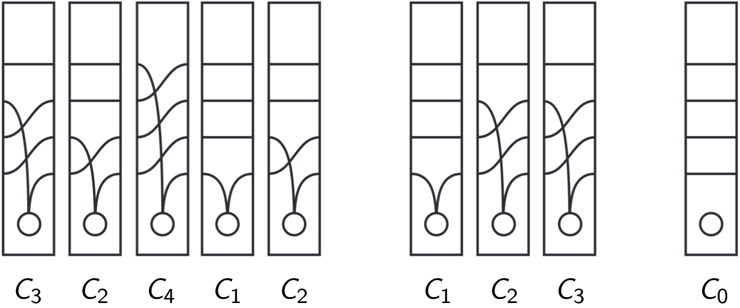
Take $b = 4$, and consider this set of five "juggling cards."



You can build any 4-ball juggling diagram from these cards.

Juggling Cards

Example: What siteswaps correspond to these card sequences?



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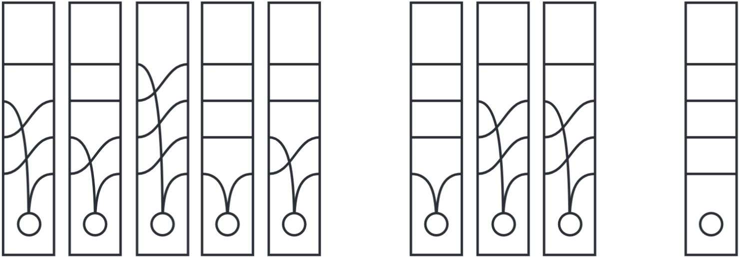

C_3 C_2 C_4 C_1 C_2 C_1 C_2 C_3 C_0

Answers: (53192), (441), ()

Counting With Cards

Theorem (Buhler, Eisenbud, Graham, & Wright, 1994)

Given an integer $n \geq 1$, there exist $(b+1)^n$ valid siteswaps with $\leq b$ balls and length n, counting repetitions and cyclic permutations separately.

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Idea of Proof: For any b, there are $b+1$ juggling cards. Each siteswap can be represented by setting n cards in a row (with repetitions possible). The total number of siteswaps will then be $(b+1)^n$.

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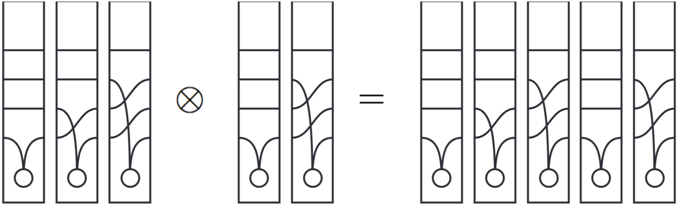
Corollary

Given an integer $n \geq 1$, there exist $(b+1)^n - b^n$ valid siteswaps with b balls and length n, counting repetitions and cyclic permutations separately.

Examples: For $b = 2$ and $n = 2$, there are $3^2 - 2^2 = 5$ valid siteswaps. For $b = 3$ and $n = 3$, there are $4^3 - 3^3 = 37$ valid siteswaps.

How to Multiply Juggling Patterns?

Juggling cards: it's easy to "concatenate" old patterns to get new ones.



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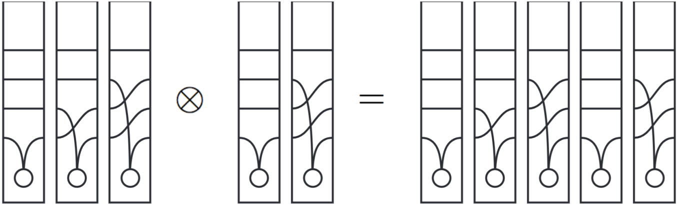

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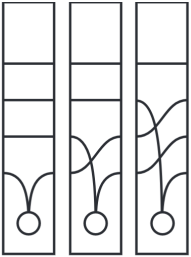
Siteswaps: direct concatenation won't always work: $(531)(51) = (53151)$, but (53151) is not a valid siteswap.

How to Multiply Juggling Patterns

Solution: Restrict to sets of "compatible" patterns.

Definition

A juggling pattern is a *ground state* pattern if there is a moment when the juggler can stop juggling, after which b "thuds" are heard as the balls hit the ground on each of the next b beats.



Ground State Siteswaps

Facts about ground state siteswaps:

- They are all compatible with the "standard" siteswap (b).
- Ground state patterns for $b = 3$: (3), (42), (423), (441), (531), (522), (6231), etc.

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- Multiplication isn't always commutative: $(3)(42) = (42)(3)$, but $(3)(42)(522) \neq (42)(3)(522)$.
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- Most ground state siteswaps can be "factored" into shorter ones: $(53403426231) = (5340)(3)(42)(6231)$.
- If a siteswap can't be factored, it is "primitive."
- The "identity" siteswap is ().

Ground State Juggling Patterns

Question: Given b, how many ground state juggling patterns are there with given length $n \geq 0$?

Theorem (Chung & Graham, 2008)

Given $b, n \geq 0$, the number of ground state juggling patterns with b balls and length n is given by

$$J_b(n) = \begin{cases} n! & \text{if } n \leq b \\ b! \cdot (b+1)^{n-b} & \text{if } n > b. \end{cases}$$

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Examples:

- $b = 3,\ n = 0$: $0! = 1$ — ()
- $b = 3,\ n = 3$: $3! = 6$ — (333), (342), (423), (441), (531), (522).
- $b = 4,\ n = 7$: $4! \cdot 5^3 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5 \cdot 5 = 3000$ siteswaps!

Proof of Chung & Graham's Theorem

- By definition, any juggling sequence $s = \{t_1, t_2, \ldots, t_n\}$ satisfies $\{t_i + i \mid 1 \leq i \leq n\} = \{b+1, b+2, \ldots, b+n\}$.
- So s corresponds to a permutation π on $\{1, 2, \ldots, n\}$, via $\pi(i) = t_i + i - b$.
- So, counting juggling sequences is the same as counting permutations that satisfy $\pi(i) = t_i + i - b \geq i - b$ for all i.

Example: Let $n = 6$, $b = 3$.

	(						)
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Choices:		1	2	3	4	4	4	

Total Count: $1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \;=\; 3!(b+1)^3 \;=\; b!(b+1)^{n-b}$.

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Theorem (τ, 2019)

Given $b \geq 4$, the number of primitive, ground state juggling patterns with b balls and length n is approximated by

$$P_b(n) \sim \frac{b+1-\rho}{|s_b'(1/\rho)|} \cdot \rho^n,$$

where $s_b(z)$ is a b-degree polynomial and ρ is a constant satisfying $0.73 \cdot \frac{1}{e^b\sqrt{b}} < 1 - \frac{\rho}{b+1} < 6.04 \cdot \frac{\sqrt{b}}{e^b}$.

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The Answer (2019): The proportion is approximately $C_b \cdot \left(\frac{\rho}{b+1}\right)^n$, i.e., the primitive siteswaps are sparse since $\frac{\rho}{b+1} < 1 - \frac{0.73}{e^b\sqrt{b}} < 0.994$:

$$\lim_{n\to\infty} C_b \cdot \left(\frac{\rho}{b+1}\right)^n < \lim_{n\to\infty} C_b \cdot (0.994)^n = 0.$$

References

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Slides online at:

https://tinyurl.com/MakingJugglingMathematical

Thank you!

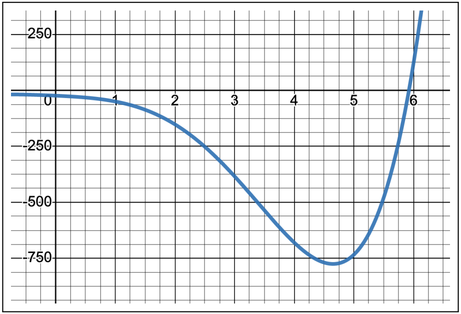
Some Future Directions

- Find improved bounds on ρ and $s_b'(1/\rho)$.
- What happens when you allow for a ball to be added or dropped (i.e., what if b can change)?
- Given a juggling siteswap s with length n, how many siteswaps of length $\leq n$ are "relatively prime" to s?
- There are *prime* siteswaps (viewed from a graph-theoretic perspective). Can we count those in a similar way?

A Problem To Play With

Use Newton's method to approximate the largest positive root ρ of

$$\bar{s}_b(x) = b! + (x - (b+1)) \sum_{k=0}^{b-1} k! \cdot x^{b-1-k}.$$



Graph of $\bar{s}_5(z)$, with $\rho \approx 5.9235$