Overview

↳ Type theory

↳ Logic

↳ Correspondence

↳ Application

Background

What is Type Theory?

↳ the study of type theories

What is a type theory?

↳ terms

↳ types

↳ A way of assigning types to terms.

Simply-Typed Lambda Calculus.

Terms :

↳ $Var = \{x, y, z, \ldots\}$ countable

↳ If M is a term and σ is a , then $\boxed{(\lambda x : \sigma . M)}$ is a term.

functions & programs!

↳ If M and N are terms, then so is $\boxed{MN}$

Types :

↳ $B = \{\alpha, \beta, \gamma, \ldots\}$ countable

↳ If σ, τ are types, then so is $\boxed{\sigma \to \tau}$

<u>OR</u> $Typ : B \mid (Typ \to Typ)$

Examples of types:

$$\alpha,\ \beta,\ \alpha \to \beta,\ \underline{\alpha \to \alpha \to \beta},\ (\alpha \to \beta) \to \alpha \to \gamma$$

Examples of terms:

$$\underbrace{x}_{},\ \underbrace{\lambda x{:}\alpha.\, x}_{\alpha \to \alpha},\ \underbrace{\lambda x{:}\alpha.\, \lambda y{:}\beta.\, y}_{\alpha \to \beta \to \beta}$$

Typing Terms:

context Γ = finite set of declarations

$$\underset{\text{Var}}{x_1} : \underset{\text{Typ}}{\sigma_1},\ x_2 : \sigma_2,\ \cdots,\ x_n : \sigma_n$$

"assumptions"

"axioms"

"Given context Γ, what terms can we type?"

$\Gamma \vdash M : \sigma$ means context Γ types M as σ.

Typing Rules

1)
$$\frac{}{\Gamma \vdash x : \sigma} \; x : \sigma \in \Gamma$$

2)
$$\frac{\Gamma \cup \{x : \sigma\} \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \to \tau}$$

3)
$$\frac{\Gamma_1 \vdash M : \sigma \to \tau \qquad \Gamma_2 \vdash N : \sigma}{\Gamma_1 \cup \Gamma_2 \vdash MN : \tau}$$

<u>Example</u> $\boxed{\lambda x{:}\sigma.\,\lambda y{:}\tau.\,y \;:\; \sigma \to \tau \to \tau}$ without assumptions.

Derivation:

$$
\dfrac{\dfrac{\dfrac{}{\{x:\sigma,\, y:\tau\} \vdash y:\tau}\; y:\tau \in \{x:\sigma,\, y:\tau\}}{\{x:\sigma\} \vdash \lambda y{:}\tau.\,y \;:\; \tau \to \tau}}{\emptyset \vdash \boxed{\lambda x{:}\sigma.\,\lambda y{:}\tau.\,y : \sigma \to \tau \to \tau}}
$$

Example $\boxed{(\lambda x{:}\sigma\to\sigma.\, x)(\lambda y{:}\sigma.\, y) : \sigma\to\sigma}$ w/o assumptions.

$$\dfrac{\dfrac{\dfrac{}{\{x:\sigma\to\sigma\}\vdash x:\sigma\to\sigma}}{\emptyset\vdash \lambda x{:}\sigma\to\sigma.\, x : (\sigma\to\sigma)\to(\sigma\to\sigma)} \qquad \dfrac{\dfrac{}{\{y:\sigma\}\vdash y:\sigma}}{\emptyset\vdash \lambda y{:}\sigma.\, y : \sigma\to\sigma}}{\emptyset\vdash (\lambda x{:}\sigma\to\sigma.\, x)(\lambda y{:}\sigma.\, y) : \sigma\to\sigma}$$

Proof Theory

↳ branch of mathematical logic

↳ studies the art of sound deductions

"proofs"

knowledge → deductions → new knowledge

"judgements"

"inference rules"

Minimal Propositional Logic

↳ judgements look like "proposition ϕ is true"

↳ $\mathrm{Prop} = \mathrm{Var} \mid (\mathrm{Prop} \to \mathrm{Prop})$

Ex. x, $x \to x$, $x \to (y \to x)$, $\phi \to \psi$.

What are the rules?

↳ rules: let $\Delta \subseteq Prop$ be a set of assumptions

1) $$\frac{}{\phi}\ \phi \in \Delta$$

2) $$\frac{\begin{array}{c}\overline{\phi}\ i \\ \vdots \\ \psi\end{array}}{\phi \to \psi}\ (\to \text{intro})_i$$

3) $$\frac{\phi \to \psi \text{ is true} \qquad \phi \text{ is true}}{\psi \text{ is true}}\ (\to \text{elim})$$

$$\dfrac{\begin{array}{c}\overline{\phi}^{\,i}\\ \vdots \\ \psi\end{array}}{\phi \to \psi}\,(\to \text{intro})_i$$

$\leftarrow$ hypothetical

Examples of Deduction (Natural Deduction)

$$\dfrac{\overline{x}^{\,0}}{x \to x}\,(\to \text{intro})_0$$

$$\dfrac{\dfrac{\overline{y}^{\,1} \quad \overline{x}^{\,0}}{y \to x}\,(\to \text{intro})_1}{x \to (y \to x)}\,(\to \text{intro})_0$$

Deductions w/ Undischarged Assumptions Δ

$$\frac{}{x}\ x \in \Delta$$

"assume x, prove x"

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y \to z}\ y \to z \in \Delta \qquad \dfrac{\dfrac{}{x \to y}\,0 \qquad \dfrac{}{x}\,1}{y}}{z}}{x \to z}\ (\to\text{intro})_1}{(x \to y) \to (x \to z)}\ (\to\text{intro})_0$$

"assume $y \to z$, prove $(x \to y) \to (x \to z)$"

Curry-Howard Correspondence

Simply-Typed Lambda Calculus $\longleftrightarrow$ Minimal Propositional Logic

↳ proposition-as-types

$x \to y \to x$ looks like $\alpha \to \beta \to \alpha$

↳ deduction-as-terms

???

Deduction of $x \to y \to x$

$$\dfrac{\dfrac{\overline{x}^{\,0} \quad \overline{y}^{\,1}}{y \to x}\,(\to\text{intro})_1}{x \to (y \to x)}\,(\to\text{intro})_0$$

Derivation of $\phi \vdash \lambda u{:}\alpha.\,\lambda v{:}\beta.\,u : \alpha \to \beta \to \alpha$

$$\dfrac{\dfrac{\overline{\{u:\alpha, v:\beta\} \vdash \boxed{u:\alpha}}}{\{u:\alpha\} \vdash \boxed{\lambda v{:}\beta.\,u : \beta \to \alpha}}}{\phi \vdash \boxed{\lambda u{:}\alpha.\,\lambda v{:}\beta.\,u : \alpha \to \beta \to \alpha}}$$

↳ types correspond to propositions

↳ during derivation, term <u>grows</u> to capture derivation

One more example. .

$$\boxed{x \to (x \to y) \to y \quad \longleftrightarrow \quad \alpha \to (\alpha \to \beta) \to \beta}$$

term: $\underline{\lambda u{:}\alpha.\ \lambda v{:}\alpha\to\beta.\ vu}$

derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{u{:}\alpha\} \vdash u{:}\alpha} \qquad \dfrac{}{\{v{:}\alpha\to\beta\} \vdash v{:}\alpha\to\beta}}{\{u{:}\alpha,\ v{:}\alpha\to\beta\} \vdash vu{:}\ \beta}}{\{u{:}\alpha\} \vdash \lambda v{:}\alpha\to\beta.\ vu{:}\ (\alpha\to\beta)\to\beta}}{\lambda u{:}\alpha.\ \lambda v{:}\alpha\to\beta.\ vu{:}\ \alpha\to(\alpha\to\beta)\to\beta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{\{u{:}\alpha\} \vdash u{:}\alpha}}\,1 \qquad \dfrac{}{\cancel{\{v{:}\alpha\to\beta\} \vdash v{:}\alpha\to\beta}}\,0}{\cancel{\{u{:}\alpha,\ v{:}\alpha\to\beta\} \vdash vu{:}\ \beta}}}{\cancel{\{u{:}\alpha\} \vdash \lambda v{:}\alpha\to\beta.\ vu{:}\ (\alpha\to\beta)\to\beta}}}{\cancel{\lambda u{:}\alpha.\ \lambda v{:}\alpha\to\beta.\ vu{:}\ \alpha\to(\alpha\to\beta)\to\beta}}$$

x $\quad x\to y$

y

$(\to\text{intro})_0$

$(x\to y)\to y$

$(\to\text{intro})_1$

$x\to(x\to y)\to y$

PANIC SLIDE

$$
\dfrac{\dfrac{\dfrac{}{\{v : x \to y\} \vdash v : x \to y} \qquad \dfrac{}{\{u : x\} \vdash u : x}}{\{u : x, v : x \to y\} \vdash uv : y}}{\{u : x\} \vdash \lambda v : x \to y.\, uv : (x \to y) \to y}
\over
\lambda u : x.\, \lambda v : x \to y.\, uv : x \to (x \to y) \to y
$$

The Correspondence, Formally

1) $$\frac{}{\phi}\ \phi \in \Delta \quad \Longrightarrow \quad \frac{}{x : \sigma}\ x : \sigma \in \Gamma$$

2) $$\frac{\begin{array}{c}\overline{\phi}^{\,0} \\ \vdots \\ \psi\end{array}}{\phi \to \psi}\ (\to \text{intro})_0 \quad \Longrightarrow \quad \frac{\Gamma \cup \{x : \sigma\} \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \to \tau}$$

3) $$\frac{\phi \to \psi \quad \phi}{\psi}\ (\to \text{elim}) \quad \Longrightarrow \quad \frac{\Gamma_1 \vdash M : \sigma \to \tau \qquad \Gamma_2 \vdash N : \sigma}{\Gamma_1 \cup \Gamma_2 \vdash MN : \tau}$$

More Type Theories

Simply-Typed λ-Calc	Min. Prop. Logic
Polymorphic Types	Second-Order Prop. Logic
Dependent Types	First-Order Logic
$\vdots$	

Application: Proof Assistants

Mathematical Statement

Mathematical Proof

$$\underbrace{\text{Term} : \text{Type}}_{\text{Proof Assistant checks type}}$$

Reference

Herman Geuvers' lecture notes: cs.ru.nl/~herman/onderwijs/provingwithCA/paper-lncs.pdf

Frank Pfenning's lecture notes: cs.cmu.edu/~fp/courses/atp/handouts/ch2-natded.pdf

Thanks for

Listening!