Shining a Rainbow-Coloured Light on the Fundamental Theorem of Algebra

by Yuveshen Mooroogen

7th of July 2021

A Polynomial of the Form $f(z) = (z-a)(z-b)(z-c)^2$

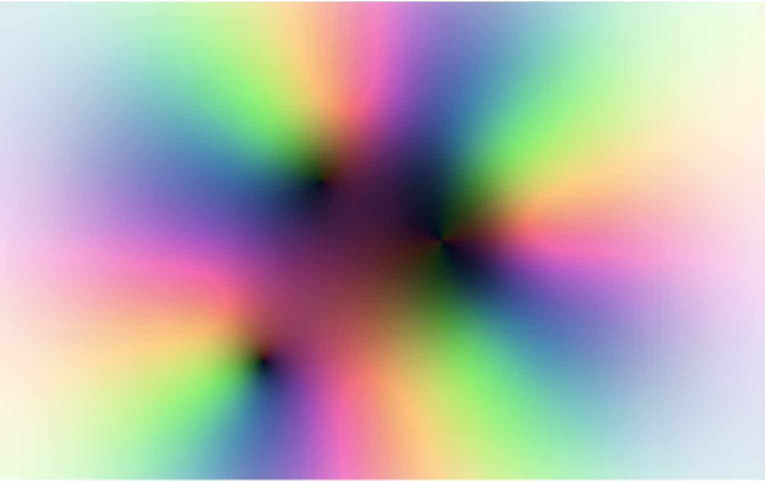


Figure: Domain colouring of the function $f(z) = (z+1+i)(z+\frac{1}{2}-\frac{1}{2}i)(z-\frac{1}{2})^2$.

The Plan

- Question 1: Why are functions of the complex numbers hard to draw?
- Question 2: How does domain colouring work?
- Question 3: What does this have to do with the FTA?

PART I

The problem with complex functions

- The **graph** of a function $f : \mathbf{R} \to \mathbf{R}$ is the set

$$\Gamma_f = \{(x, f(x)) : x \in \mathbf{R}\}.$$

- This is a subset of $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$.

Figure: Graph of the function $f(x) = x^2$, from Wolfram|Alpha.

Real-Valued Functions of Two Real Variables

- The **graph** of a function $f : \mathbf{R}^2 \to \mathbf{R}$ is the set

$$\begin{aligned} \Gamma_f &= \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathbf{R}^2\} \\ &\cong \{(x, y, f(x, y)) : x, y \in \mathbf{R}\}. \end{aligned}$$

- This is a subset of $\mathbf{R}^2 \times \mathbf{R} = (\mathbf{R} \times \mathbf{R}) \times \mathbf{R} = \mathbf{R}^3$.

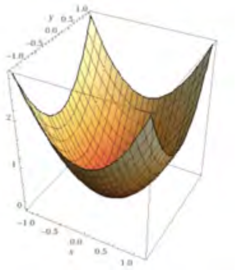


Figure: Graph of the function $f(x, y) = x^2 + y^2$, from Wolfram|Alpha.

Complex-Valued Functions of One Complex Variable

- The **graph** of a function $f : \mathbf{C} \to \mathbf{C}$ is the set

$$\begin{aligned} \Gamma_f &= \{(\mathbf{z}, f(\mathbf{z})) : \mathbf{z} \in \mathbf{C}\} \\ &\cong \{(x, y, \mathbf{f}(\mathbf{z})) : x, y \in \mathbf{R}\} \\ &\cong \{(x, y, f_1(x, y), f_2(x, y)) : x, y \in \mathbf{R}\}. \end{aligned}$$

- This is a subset of $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C} \cong (\mathbf{R} \times \mathbf{R}) \times (\mathbf{R} \times \mathbf{R}) = \mathbf{R}^4$.
- Moral: We need to do something clever to draw these functions!

Notation

- The **complex numbers C** are the set of all ordered pairs of real numbers (x, y), on which we define three operations:
  1. Multiplication: $(x, y) \cdot (u, v) = (xu - yv, xv + yu)$.
  2. Addition: $(x, y) + (u, v) = (x + u, y + v)$.
  3. Scalar multiplication: $u \cdot (x, y) = (ux, uy)$.

- The first two operations give **C** the structure of a **field**, the last two equip it with a **two-dimensional real vector space** structure.

- We write $z = x + iy$ for the complex number (x, y). Thus, each z identifies a unique point on the **complex plane.**

Notation

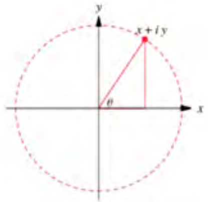


Figure: The Complex Plane, from Wolfram MathWorld

- The **modulus** $|z|$ of a complex number z is its distance from the origin. The **argument** $\arg(z)$ of z is the angle that the segment joining it to the origin makes, relative to the positive real axis.

- These quantities allow us to write z in **modulus-argument form** (or **polar coordinate form**) as $(|z|, \arg(z))$.

PART II

Something clever

A Recipe for Domain Colouring: The Case of $f(z) = z^3$

- Step 1: Consider complex planes for the domain and codomain of f.

Figure: Domain and codomain of f.

- A point w in the codomain may be described by an ordered pair of real numbers.

A Recipe for Domain Colouring: The Case of $f(z) = z^3$

- Step 2: Impose a shaded colour wheel on the codomain.

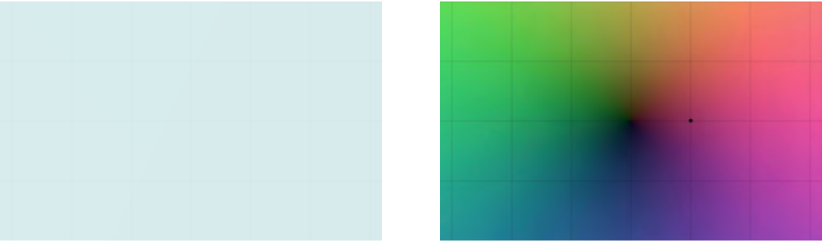


Figure: Shaded colour wheel on the codomain.

- We may now describe w by the pair $(\text{Shade}(w), \text{Hue}(w))$, where

$$\text{Shade}(w) = |w| \quad \text{and} \quad \text{Hue}(w) = \arg(w).$$

- Step 3: Colour the set $f^{-1}(\{w\})$ with the same hue and shade as w.

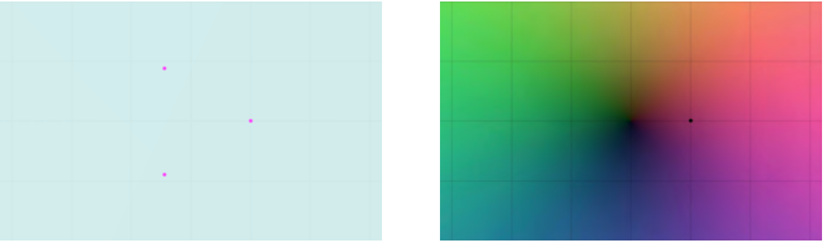


Figure: Colouring of the set $f^{-1}(1)$ in the domain.

- A coloured point $z = x + iy$ in $f^{-1}(1)$ now gives us four pieces of information:

$$\begin{aligned} z &= (x, y, \text{Shade}(w), \text{Hue}(w)) \\ &= (x, y, \text{Shade}(f(z)), \text{Hue}(f(z))) \end{aligned}$$

A Recipe for Domain Colouring: The Case of $f(z) = z^3$

- Step 4: Apply this colouring rule to all points of the codomain.

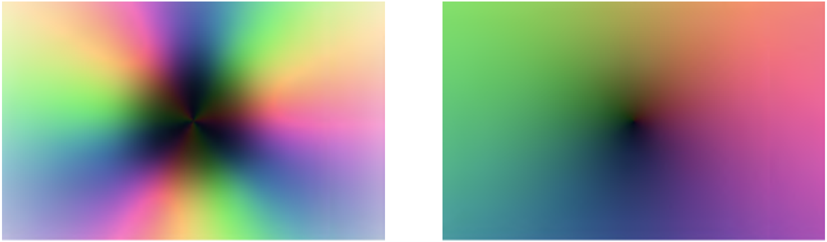


Figure: Domain colouring of the function $f(z) = z^3$.

Example 1: The Identity Function

- As $\mathrm{id}^{-1}(w) = \{w\}$, the domain colouring is the chosen colour wheel.

Figure: Domain colouring of the function $\mathrm{id}(z) = z$.

- Since $f^{-1}(w) = \mathbf{C}$ or $\varnothing$, the domain colouring is monochromatic.

Figure: Domain colouring of the function $f(z) = 2 - i$.

Example 3: Monomials

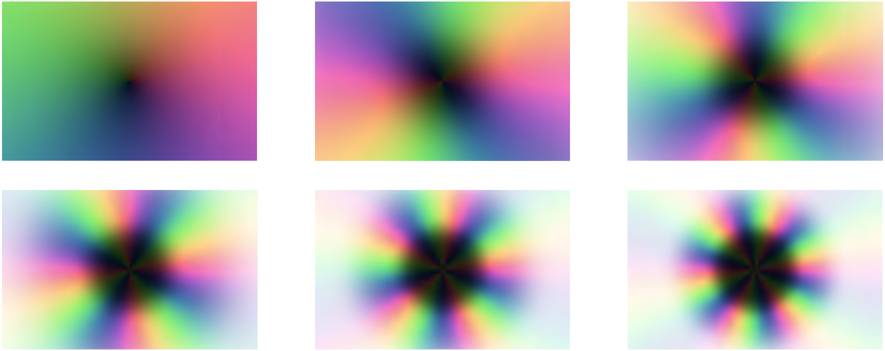


Figure: Domain colouring of the functions $f_n(z) = z^n$ for n = 1, . . . , 6.

Example 3: Monomials

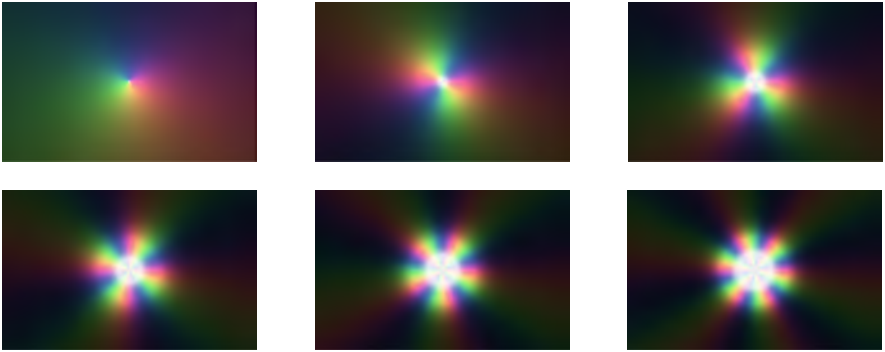


Figure: Domain colouring of the functions $f_n(z) = \frac{1}{z^n}$ for n = 1, . . . , 6.

Example 4: Polynomials

- Proposition 1: The behaviour of a polynomial of degree n is dominated by z^n as $|z| \to \infty$.

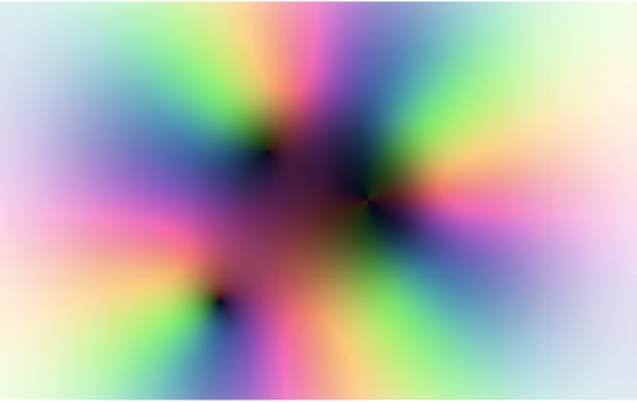


Figure: Domain colouring of a polynomial of the form $f(z) = (z-a)(z+b)(z-c)^2$.

Example 4: Polynomials

- Proposition 2: Near a root of multiplicity n, a polynomial behaves like z^n does near the origin.

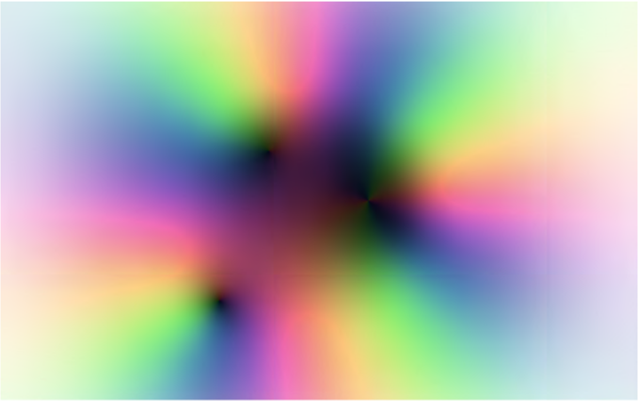


Figure: Domain colouring of a polynomial of the form $f(z) = (z-a)(z+b)(z-c)^2$.

Something for you to play with!

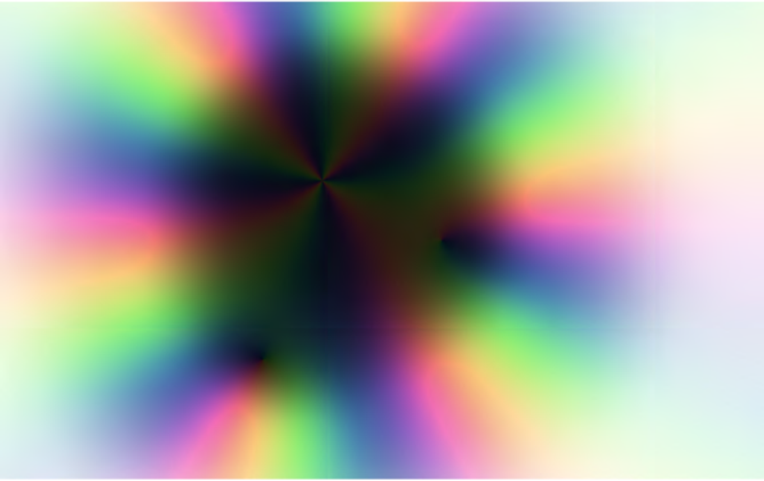


Figure: What kind of a polynomial is this?

PART III

d'Alembert's proof (1746), colourised by Velleman (2015)

Statement

- Fundamental Theorem of Algebra: Any nonconstant single-variable polynomial with complex coefficients has a root in $\mathbf{C}$.

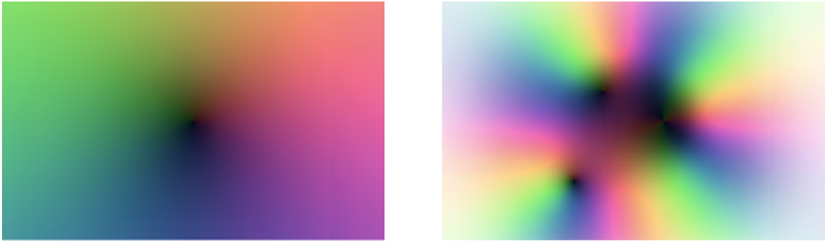
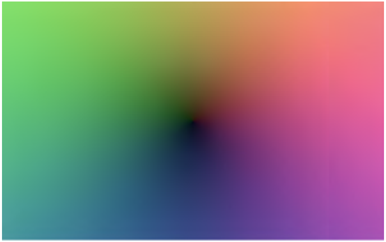


Figure: Two nonconstant polynomials.

Statement

- Fundamental Theorem of Algebra: The domain colouring of any nonconstant single-variable polynomial with complex coefficients contains a black point.



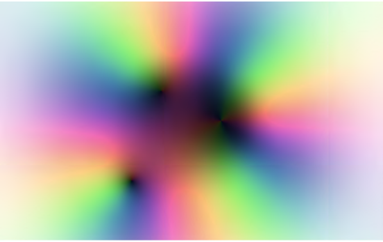


Figure: Two nonconstant polynomials.

d'Alembert's Lemma (Super Important)

- Darker Neighbourhood Principle: If f is a nonconstant polynomial and z is a point such that $f(z) \neq 0$, then for every $\epsilon > 0$, there is a z_{darker} with $|z - z_{\text{darker}}| < \epsilon$ and $|f(z_{\text{darker}})| < |f(z)|$.

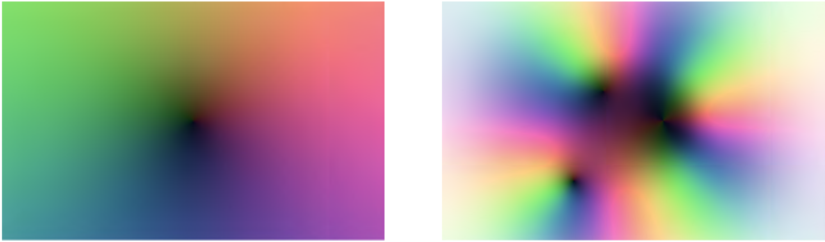
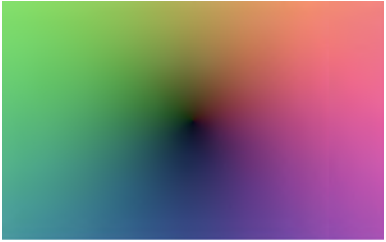


Figure: Two nonconstant polynomials.

d'Alembert's Lemma (Super Important)

- Darker Neighbourhood Principle: Let z be a point in the domain colouring of a nonconstant polynomial. If z is not black, then every disc centred at z contains a strictly darker point z_{darker}.



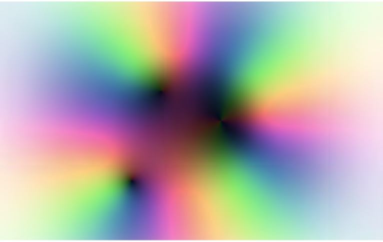


Figure: Two nonconstant polynomials.

Proof of the Fundamental Theorem of Algebra

- Assume that f is a nonconstant polynomial. We will show that its domain colouring contains a black point.

Proof of the Fundamental Theorem of Algebra

- Step 1: Choose a large square $S = [-R, R] \times [-R, R]$ in the domain of the function f.

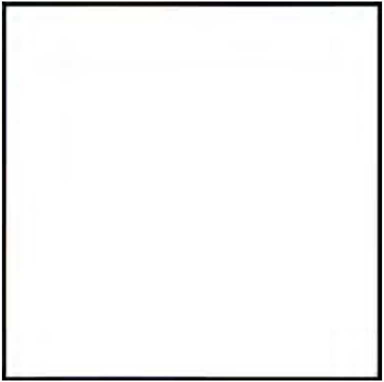


Figure: Sketch of the portion of the domain colouring of f.

Proof of the Fundamental Theorem of Algebra

- Near the boundary, f behaves like its highest-degree term.

Figure: The colours get lighter as we move outside the white square.

- Step 2: Observe that, by the Extreme Value Theorem, the function $|f(z)|$ achieves a minimum at a point z_{darkest} on this square.

Figure: Since f gets lighter near the boundary, z_{darkest} cannot be on the boundary of S.

Proof of the Fundamental Theorem of Algebra

- Step 2: Observe that, by the Extreme Value Theorem, the function $|f(z)|$ achieves a minimum z_{darkest} on this square.

Figure: Since f gets lighter near the boundary, z_{darkest} is in the interior of S.

Proof of the Fundamental Theorem of Algebra

- Step 3: If z_{darkest} is not black, then by the Darker Neighbourhood Principle, there is a strictly darker point nearby.

Figure: Consider a disc D centred at z_{darkest}.

Proof of the Fundamental Theorem of Algebra

- Step 3: If z_{darkest} is not black, then by the Darker Neighbourhood Principle, there is a strictly darker point nearby.

Figure: By the Darker Neighbourhood Principle, D contains a darker point z_{darker}.

Proof of the Fundamental Theorem of Algebra

- But this would contradict that z_{darkest} is the darkest point on S!

Figure: By the Darker Neighbourhood Principle, D contains a darker point z_{darker}.

Proof of the Fundamental Theorem of Algebra

- Thus, z_{darkest} is black.

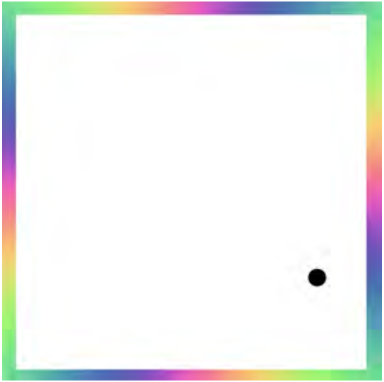


Figure: Q.E.D.

References and Suggested Reading

References and Suggested Reading

- Reusser, R. *Domain Coloring for Complex Functions.* (2018) Accessed at `https://observablehq.com/@rreusser/domain-coloring-for-complex-functions` on 31.07.20
- Wolfram|Alpha. *Graph of* $f(x) = x^2$. Accessed at `https://www.wolframalpha.com/input/?i=graph+x%5E2` on 28.07.20.
- Wolfram|Alpha. *Graph of* $f(x, y) = x^2 + y^2$. Accessed at `https://www.wolframalpha.com/input/?i=graph+3d+plot` on 28.07.20.
- Weisstein, E. W. *Complex Number.* From MathWorld–A Wolfram Web Resource. Accessed at `https://mathworld.wolfram.com/ComplexNumber.html` on 28.07.20.

References and Suggested Reading

- Lundmark, H. *Visualizing Complex Analytic Functions Using Domain Coloring*. (2017) Accessed at `https://users.mai.liu.se/hanlu09/complex/domain_coloring.html` on 25.07.20.
- Velleman, D.J. The Fundamental Theorem of Algebra: A Visual Approach. *Math Intelligencer* **37**, 12-21 (2015). `https://doi.org/10.1007/s00283-015-9572-7`.
- Stein, E.M., and Shakarchi, R. *Complex Analysis.* Princeton University Press, 2003.

Thank you!

yuveshen.mooroogen@mail.utoronto.ca