A29 Wk 10a

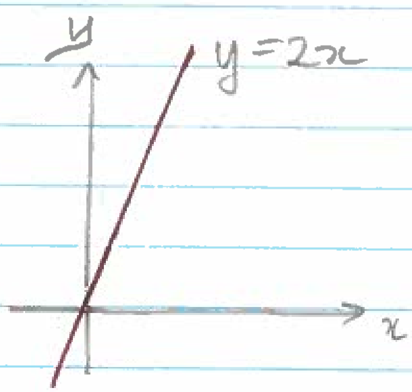
This week: §5.1 – 5.3 Integration and Area.

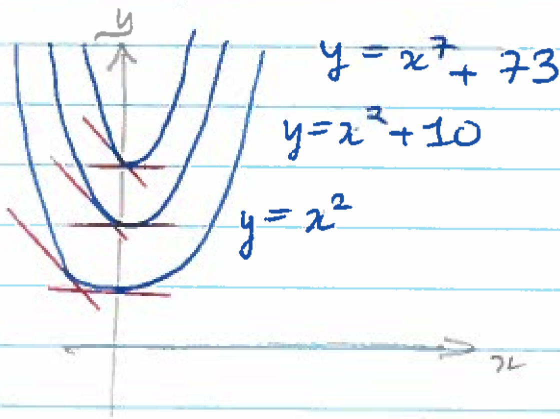
Recall,

differentiation: function $\rightsquigarrow$ slope
$f(x)$ $\qquad$ $f'(x)$

The reverse process is:

anti-differentiation: slope $\rightsquigarrow$ function
$f'(x)$ $\qquad$ $f(x)$





<u>Fact</u>: If $F'(x) = G'(x)$ are two functions with the same derivative then

$$F(x) = G(x) + C$$

for some constant C.

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[G(x) + C].$$

<u>Def^n</u>: If $F'(x) = f(x)$ then $F(x)$ is an ANTI-DERIVATIVE of $f(x)$.

<u>Ex</u>: Find an anti-derivative of $f(x) = 2x$.

\# want $F(x)$ so that $F'(x) = 2x$

We take $F(x) = x^2$.

<u>Ex</u>: Find an anti-derivative of $f(x) = \cos(2x)$

\# want $F(x)$ so that $F(x) = \cos(2x)$

$$\begin{cases} \text{Guess: } F(x) = \sin(2x) \\ \text{Check: } F'(x) = \dfrac{d}{dx}\left[\sin(2x)\right] = \dfrac{d\sin(2x)}{d(2x)} \cdot \dfrac{d(2x)}{dx} \end{cases}$$

$$= 2\cos(2x)$$

$$\begin{cases} \text{Guess: } F(x) = \frac{1}{2}\sin(2x) \\ \text{Check: } F'(x) = \dfrac{d}{dx}\left[\frac{1}{2}\sin(2x)\right] = \frac{1}{2} \cdot 2\cos(2x) \end{cases}$$

$$= \cos(2x).$$

Thus we pick $F(x) = \frac{1}{2}\sin(2x)$.

Integration (the process of finding anti-deriv.s)

Defn :

integrand

$$\int f(x)\,dx = g(x) + C$$ ← constant of integration

"the (indefinite) integral of $f(x)$ is $g(x)$"

means $\frac{d}{dx}\left[g(x)+c\right] = f(x)$

NB : $\int f(x)\,dx$ is an anti-derivative of $f(x)$.

Ex : calculate $\int x^2\,dx$.

Guess: $g(x) = x^3$

Check: $g'(x) = 3x^2$

Guess : $g(x) = \frac{1}{3}x^3$

Check: $g'(x) = \frac{1}{3}\cdot 3 \cdot x^2 = x^2$

Thus, $\int x^2\,dx = \frac{1}{3}x^2 + C$.

<u>Thm</u>: $\int a f(x) + b g(x)\, dx = a\int f(x) + b\int g(x)\, dx$

Pf: Suppose $F'(x) = f(x)$ and $G'(x) = g(x)$.

F is an antideriv of f(x)

G is an antideriv of g(x)

We then have

$$\frac{d}{dx}\left[a F(x) + b G(x)\right]$$

$$= a F'(x) + b G'(x)$$

$$= a f(x) + b g(x)$$

$aF(x) + bG(x)$ is an anti-deriv of $af(x) + bg(x)$.

<u>Ex</u>: Find $\int 1 + 2x\, dx$

$$= \int 1\, dx + \int 2x\, dx$$

$$= x + x^2 + C$$

Check: $\frac{d}{dx}\left[x + x^2 + C\right] = 1 + 2x + 0 = 1 + 2x$ ✓.

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Integrating Polynomials

Fact: If $k \neq -1$ then $\int x^k dx = \frac{1}{k+1} x^{k+1} + C$.

Pf: $\frac{d}{dx}\left[\frac{1}{k+1} x^{k+1}\right] = \frac{k+1}{k+1} x^{k+1-1} = x^k$ ✓.

Ex: Find $\int 3x^2 + \frac{1}{7}x + \frac{2}{3}\, dx$

$$= \int 3x^2 dx + \int \frac{1}{7} x\, dx + \int \frac{2}{3} dx$$

$$= x^3 + \frac{1}{2 \cdot 7} \cdot x^2 + \frac{2}{3}x + C.$$

Integrating Exponentials

Ex: Find $\int e^{5x} dx$

$\begin{cases} \text{Guess: } F(x) = e^{5x} \\ \text{Check: } F'(x) = 5e^{5x} \end{cases}$

$\begin{cases} \text{Guess: } F(x) = \frac{1}{5} e^{5x} \\ \text{Check: } F'(x) = \frac{1}{5} \cdot 5 \cdot e^{5x} = e^{5x}. \end{cases}$

Thus, $\int e^{5x} dx = \frac{1}{5} e^{5x} + C$.

In general, $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$

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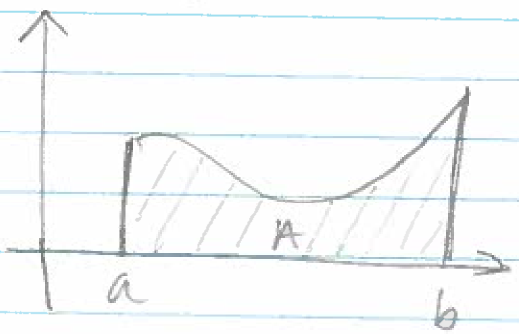
Area and Rectangles

Our major application of integration will be finding the area under a curve.

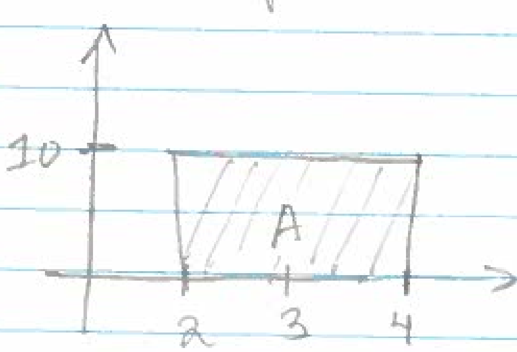


"How do we calculate A?"

First we try the following simpler idea

Ex: Find the area under $f(x) = 10$ for $x \in [2,4]$

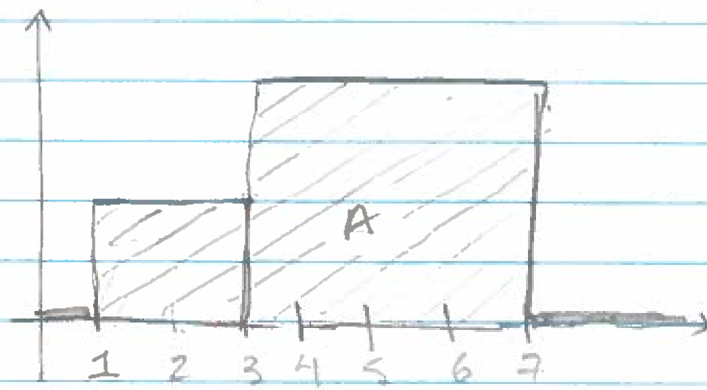
Draw a picture.



$A = l \cdot w = 10(4-2)$
$= 20.$

Ex: Find the area under $f(x) = \begin{cases} 5 & 1 \leq x \leq 3 \\ 10 & 3 < x \leq 7 \\ 0 & \text{otherwise} \end{cases}$

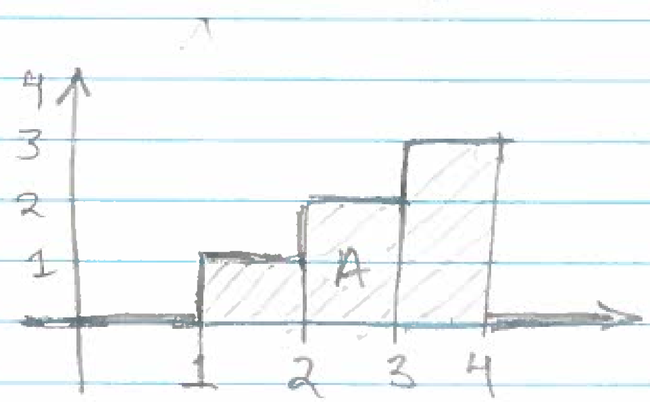
Draw a picture



$$A = 5(3-2) + 10(7-3)$$
$$= 5 + 10 \cdot 4 = 45.$$

Ex: Find the area under $f(x) = \begin{cases} 1 & 1 \leq x < 2 \\ 2 & 2 \leq x < 3 \\ 3 & 3 \leq x < 4 \\ 0 & \text{otherwise} \end{cases}$

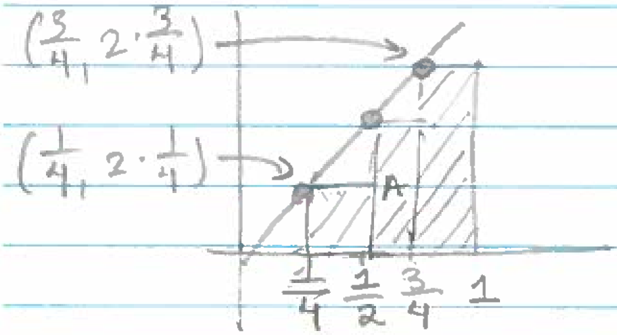
Draw the picture



$$A = 1(2-1) + 2(3-2) + 3(4-3)$$
$$= 1 + 2 + 3 = 6.$$

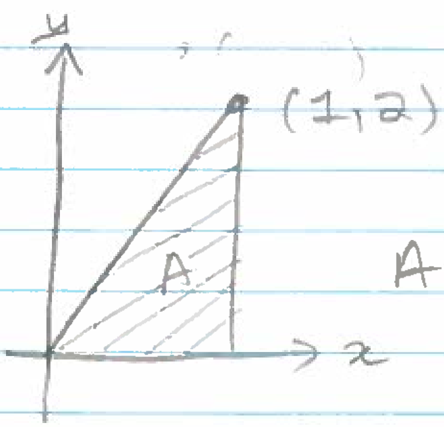
Ex: Approximate the area underneath $y=2x$ between $x=0$ and $x=1$ by dividing into rectangles at

$$x=\frac{1}{4} \quad x=\frac{1}{2} \quad x=\frac{3}{4}$$



$$\begin{aligned} A &= \left(\tfrac{1}{2}-\tfrac{1}{4}\right)\cdot\left(2\cdot\tfrac{1}{4}\right) \\ &\quad + \left(\tfrac{3}{4}-\tfrac{1}{2}\right)\left(2\cdot\tfrac{1}{2}\right) \\ &\quad + \left(1-\tfrac{3}{4}\right)\cdot\left(2\cdot\tfrac{3}{4}\right) \\ &= \tfrac{1}{4}\left(2\cdot\tfrac{1}{4}\right)+\tfrac{1}{4}\left(2\cdot\tfrac{1}{2}\right)+\tfrac{1}{4}\left(2\cdot\tfrac{3}{4}\right) \\ &= \tfrac{1}{4}\left(\tfrac{1}{2}+1+\tfrac{3}{2}\right) = \tfrac{1}{4}(3) = \tfrac{3}{4}. \end{aligned}$$

We note that the correct area is:



$$A = \frac{1}{2}\cdot 1\cdot 2 = 1.$$

Exercise: Add rectangles.

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Summation Notation

Notation: $\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \dots + f(b)$

"the sum of $f(i)$ from $i=a$ to $i=b$"

Ex: Write out the sum:

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

Ex: Write out the sum

$$\sum_{n=10}^{15} 2n = 2\cdot 10 + 2\cdot 11 + 2\cdot 12 + 2\cdot 13 + 2\cdot 14 + 2\cdot 15$$

Ex: Write the area under the graph

$$f(x) = \begin{cases} 1 & 0 < x \le 1 \\ 1/2 & 1 < x \le 2 \\ 1/4 & 2 < x \le 3 \\ 1/8 & 3 < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Express area with n equal parts as a sum.

using summation notation.

$$A = 1\cdot(1-0) + \frac{1}{2}(2-1) + \frac{1}{4}(3-2) + \frac{1}{8}(4-3)$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \sum_{i=0}^{3} \frac{1}{2^i}$$