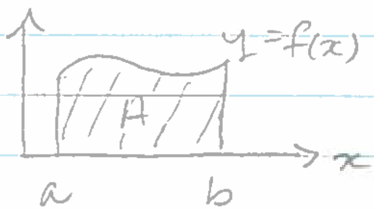
A29 Wk 11a

This week:

The Fundamental Theorem of Calculus.
Riemann Sums
Integration by Substitution

Also — Course evaluations. Please do them!

Recall,



$$= \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)(x_{k+1}-x_k)$$

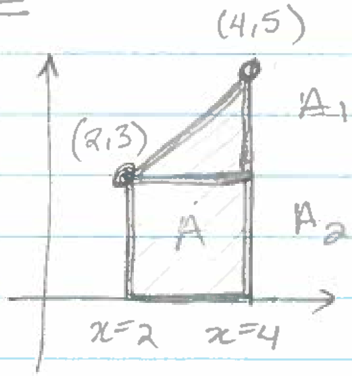
$a = x_1 < x_2 < \dots < x_n = b$

AREA DEFINITE INTEGRAL RIEMANN SUM

Also, $1+2+\dots+n = \frac{n(n+1)}{2}$.

Ex: Find the area under $y=1+x$ between $x=2$ and $x=4$
In three ways: ① geometrically.
② By integration.
③ As a Riemann sum.

Soln ①



$A_1 = \frac{1}{2}(4-2)(5-3)$
$= 2$

$A_2 = 3\cdot(4-2)$
$= 6$

Thus, $A = 6+2 = 8$.

Soln ②

$A = \int_2^4 1+x\,dx$

$F(x) = x + \frac{1}{2}x^2 \Rightarrow F'(x) = 1+2x$

$A = \left[4 + \frac{1}{2}\cdot 4^2\right] - \left[2 + \frac{1}{2}\cdot 2^2\right]$
$= 4+8-2-2$

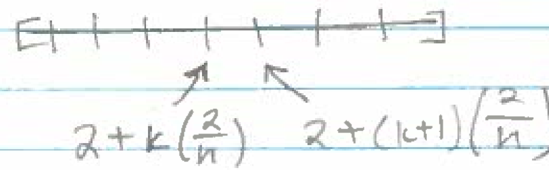
Ex (con't)

Solu ③ # Divide $[2,4]$ into n equally sized parts

$$x_0 = 2$$

$$x_1 = 2 + \frac{4-2}{n} = 2 + \frac{2}{n}$$

$$x_2 = 2 + \frac{2}{n} + \frac{2}{n}$$



$$x_3 = 2 + \frac{2}{n} + \frac{2}{n} + \frac{2}{n}$$

$$x_n = 2 + n\left(\frac{2}{n}\right) = 2+2 = 4.$$

Write out the Riemann sum.

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\left[x_{k+1} - x_k\right]$$

n

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(2+k\frac{2}{n}\right)\left[\left(2+(k+1)\left(\frac{2}{n}\right)\right) - \left(2+k\left(\frac{2}{n}\right)\right)\right]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(2+k\left(\frac{2}{n}\right)\right)\cdot\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f\left(2+k\left(\frac{2}{n}\right)\right) = \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} 1+\left(2+k\left(\frac{2}{n}\right)\right)$$

Ex (con't)

Evaluate the Riemann sum

$$A = \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} 1 + \left(2 + k\left(\frac{2}{n}\right)\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} 3 + k\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{2}{n}\sum_{k=1}^{n} 3 + \frac{4}{n^2}\sum_{k=1}^{n} k\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\cdot(3n) + \frac{4}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} 6 + 2\,\frac{n^2+n}{n^2} = 6+2 = 8.$$

Summary:

Divide interval in to n equally sized parts

Write out the Riemann sum (Simplify!)

Evaluate Riemann sum.

A29 Wk 11a

We need another summation formula:

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

We check that this always holds.

$$1^2=\frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}=\frac{1\cdot 2\cdot 3}{6}=1.$$

Suppose it holds for k. That is,

$$\sum_{i=1}^{k} i^2=\frac{k(k+1)(2k+1)}{6}.$$

We get

$$(k+1)^2+\sum_{i=1}^{k} i^2$$

$$=(k+1)^2+\frac{k(k+1)(2k+1)}{6}$$

$$=(k+1)\left[(k+1)+\frac{k(2k+1)}{6}\right]$$

$$=(k+1)\left[\frac{6(k+1)}{6}+\frac{k(2k+1)}{6}\right]$$

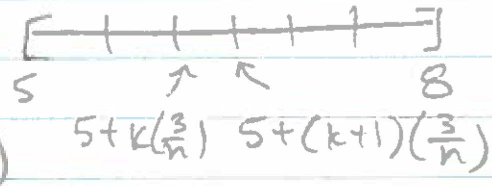
$$=(k+1)\left[\frac{6k+6+2k^2+k}{6}\right]=\frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Ex: Use a Riemann sum to find the area under $y=x^2$ from $x=5$ to $x=8$.

Split the interval into n equal parts.

$x_0 = 5 \quad x_1 = 5 + \frac{8-5}{n} = 5 + \frac{3}{n} \quad x_2 = 5 + 2\left(\frac{3}{n}\right)$

Write out the Riemann sum



$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)(x_{k+1} - x_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(5 + k\left(\frac{3}{n}\right)\right)\left[\frac{3}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{3}{n}\right] \sum_{k=1}^{n} \left(5 + k\left(\frac{3}{n}\right)\right)^2$$

$$= \lim_{n\to\infty} \left[\frac{3}{n}\right] \sum_{k=1}^{n} 25 + 10\cdot k\left(\frac{3}{n}\right) + k^2\left(\frac{9}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{75}{n}\sum_{k=1}^{n} 1 + \frac{90}{n^2}\sum_{k=1}^{n} k + \frac{27}{n^3}\sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{75}{n}\cdot n + \frac{90}{n^2}\cdot\frac{n(n+1)}{2} + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

Evaluate the Riemann sum

$$A = 75 + 45 + 9 = 129$$

Proving the FTC

Thm: If f is continuous and

$$A(x) = \int_0^x f(t)\,dt$$

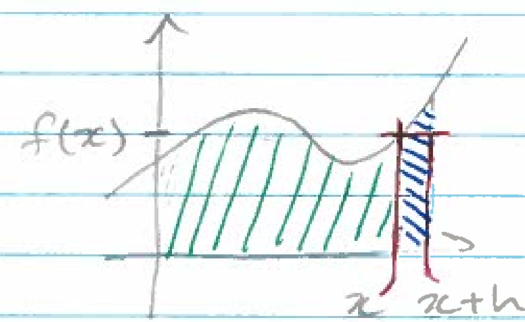
then $A'(x) = f(x)$.

Pf:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$\approx \lim_{h \to 0} \frac{f(x) \cdot h}{h}$$

$$= f(x)$$



NB: The area of the blue region is approximately:

$$f(x) \cdot ((x+h) - x)$$
$$= f(x) \cdot h$$

The Fundamental Theorem

If $f(x)$ is continuous ad $F'(x) = f(x)$
then
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Pf:

Define $A(x) = \int_a^x f(t)\,dt$

We have $A'(x) = f(x)$ and $F'(x) = f(x)$.

Thus $A(x) = F(x) + C$.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$
$$= A(b).$$
$$= A(b) - A(a) \quad \# \; A(a) = 0$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

A29 Wk 11b

Thm: If $a < c < b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Pf: By the Fundamental Theorem

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= [F(\cancel{c}) - F(a)] + [F(b) - F(\cancel{c})]$$

$$= F(b) - F(a)$$

Summary of Integrals (0) $\int_a^b f(x)\,dx$ is SIGNED area

(1) $\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$

(2) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Integration by Substitution.

"How do we integrate $\int 2xe^{x^2}\,dx$?"

Use the chain rule backwards!

Recall, $\frac{d}{dx} f(y) = f'(y) \frac{dy}{dx}$.

Ex: $\int 2x e^{x^2} dx$

$$\boxed{\begin{aligned} \text{Let } \; y &= x^2 \\ \frac{dy}{dx} &= 2x \end{aligned}}$$

$= \int \frac{dy}{dx} \cdot e^{y} dx$

$= \int e^{y} dy$

$= e^{y} + c = e^{x^2} + c.$

Ex: $\int \frac{3x^2}{1+x^3} dx$

$$\boxed{\begin{aligned} \text{Let } \; y &= x^3 \\ \frac{dy}{dx} &= 3x^2 \end{aligned}}$$

$= \int \frac{\frac{dy}{dx}}{1+y} dx$

$= \int \frac{1}{1+y} dy$

$= \ln|1+y| + c$

$= \ln|1+x^3| + c$

Ex: $\int \frac{1}{\sqrt{x}} \sin(\sqrt{x})\, dx$

$= \int \frac{2}{2\sqrt{x}} \sin(\sqrt{x})\, dx$ # introduce the terms needed to complete dy

$= 2\int \frac{1}{2\sqrt{x}} \sin(\sqrt{x})\, dx$

$= 2\int \frac{dy}{dx} \sin(y)\, dx$

let $y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$= 2\int \sin(y)\, dy$

$= 2\left[-\cos(y)\right] + c$

$= -2\cos(\sqrt{x}) + c$

Notation: we write

$$dy = \frac{dy}{dx} \cdot dx$$

and make substitutions accordingly.

Ex: $\int x\cos(x^2)\, dx$

$u = x^2$

$du = 2x\, dx$

$= \frac{1}{2}\int 2x\cos(x^2)\, dx$

$= \frac{1}{2}\int \cos(u)\, du = \frac{1}{2}\sin(u) + c = \frac{1}{2}\sin(x^2) + C.$