This week:

§2.1 ∘ limits
2.2 ∘ continuity
2.3, 2.4 ∘ rates of change

Today:

- limits
- continuity

Limits

Limits allow us to examine the local behaviour of a function.

They answer the question:

"When x is close to a what is $f(x)$ close to?"

Ex: When x is close to 3 what is $f(x) = 2x+3$ close to?

Make a table of values.

x	$f(x)$
2	7.0
2.9	8.8
2.99	8.98
3.01	9.02
3.1	9.2

We conclude $f(x) = 2x+3$ is near $2 \cdot 3 + 3 = 9$.

There are more subtle kinds of limits than this.

$\boxed{\text{We will not ask you to perform this kind of calculation.}}$

Mat A29 wk 2a

Ex: When x is close to 0 what is $\frac{\sin(x)}{x}$ close to?

Make a table of values

x	$\frac{\sin(x)}{x}$ (rad!)
-0.1	0.998...
-0.01	0.999983...
-0.001	0.99999983...
0.001	0.99999983...
0.01	0.999983...
0.1	0.998...

We conclude $\frac{\sin(x)}{x}$ is close to 1.

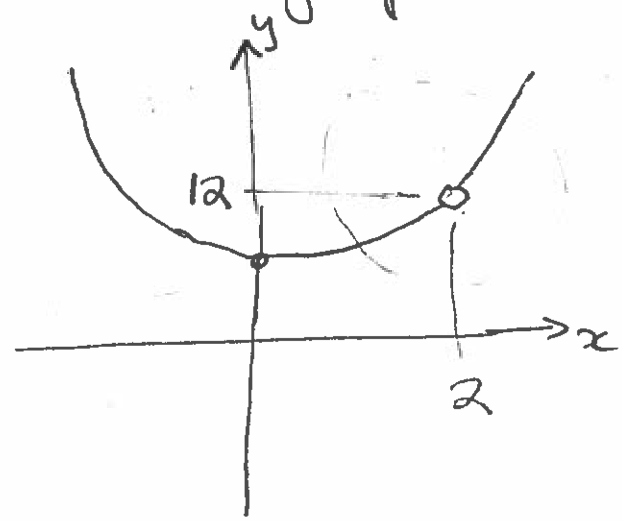
We write:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Please memorize

Ex: When x is close to 2 what is $f(x) = \frac{x^3 - 8}{x - 2}$ close to?

Draw a graph

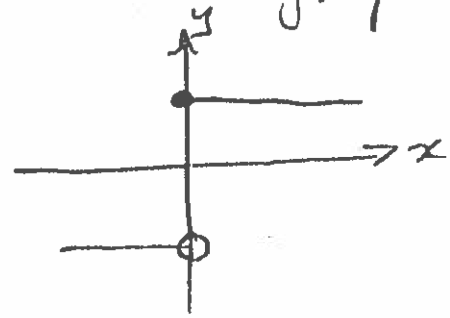


observe that:

$$\lim_{x \to 2} f(x) = 12.$$

Ex: When x is close to 0 what is $f(x) = \frac{|x|}{x}$ close to?

Draw the graph



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Examples of Limits

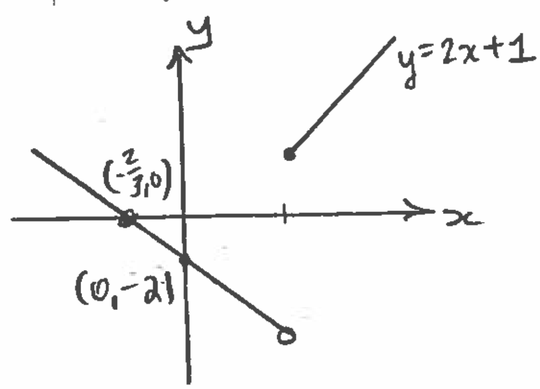
Ex: consider

$$f(x) = \begin{cases} 2x+1 & x \geq 1 \\ -3x-2 & x < 1. \end{cases}$$

What are:

$\lim\limits_{x \to 1^+} f(x)$ and $\lim\limits_{x \to 1^-} f(x)$?

graph the function



We see that:

$$\lim_{x \to 1^-} f(x) = -5$$

and $$\lim_{x \to 1^+} f(x) = 3$$

Thus, $\lim\limits_{x \to 1} f(x)$ does not exist.

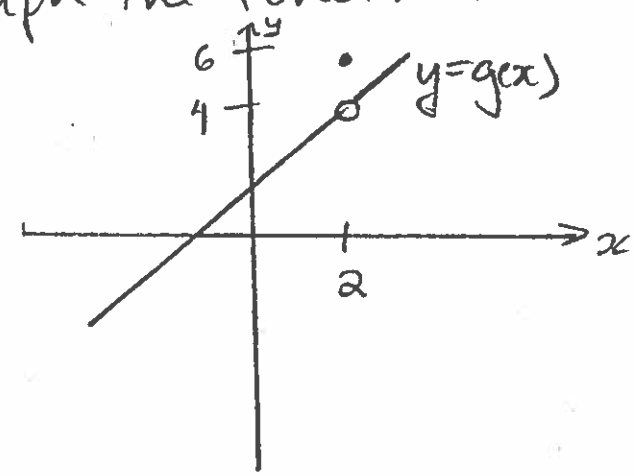
Ex: consider

$$g(x) = \begin{cases} 6 & x = 2 \\ \dfrac{x^2-4}{x-2} & x \neq 2 \end{cases}$$

What are

$\lim\limits_{x \to 2^+} g(x)$ and $\lim\limits_{x \to 2^-} g(x)$?

Graph the function



We see that.

$$\lim_{x \to 2^+} g(x) = 4$$

$$\lim_{x \to 2^-} g(x) = 4.$$

Thus,

$\lim\limits_{x \to 2} g(x) = 4$ (and not 6

Defn:

- $\lim_{x \to c^-} f(x) = L$

  "As x increases to c $f(x)$ approaches L"

- $\lim_{x \to c^+} f(x) = M$

  "As x decreases to c $f(x)$ approaches M"

- $\lim_{x \to c} f(x) = N$

  "When x is near c $f(x)$ is near N."

Fact:

$\lim_{x \to a} f(x) = L$

is equivalent to

$\left(\lim_{x \to a^-} f(x) = L\right)$ and $\left(\lim_{x \to a^+} f(x) = L\right)$

The Limit Principles (p86)

Let c be a number.

$\lim_{x \to a} f(x) = L$

$\lim_{x \to a} g(x) = M$

L1) $\lim_{x \to a} c \cdot f(x) = c \cdot L$

L2) $\lim_{x \to a} (f(x) + g(x)) = L + M$

$\lim_{x \to a} (f(x) - g(x)) = L - M$

L3) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

L4) If $M \neq 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$$

L5) If H is continuous at $x = L$ then

$$\lim_{x \to a} H(f(x)) = H(L)$$

continuity

Defn:

A function $f(x)$ is CONTINUOUS at a point $x=c$ if

$$\lim_{x \to c} f(x) = f(c)$$

Ex:

The function $f(x) = |x|$ is continuous at $x=0$.

Make a table of values.

x	$f(x) = \lvert x \rvert$
-0.001	0.001
-0.01	0.01
-0.1	0.1
0.1	0.1
0.01	0.01
0.001	0.001

Thus $\lim_{x \to 0} |x| = |0| = 0$.

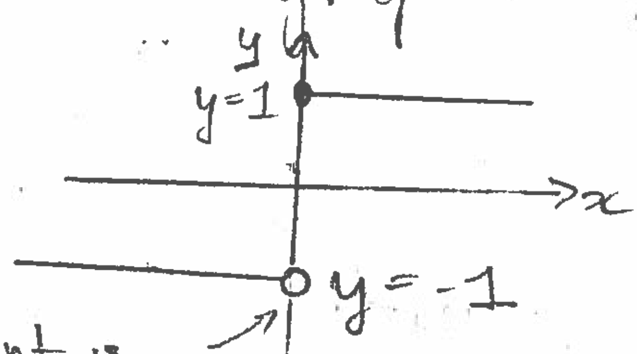
Ex: The function $g(x) = \dfrac{x}{|x|}$ is NOT continuous at $x=0$.

Recall, $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$

Make a table of values.

x	$g(x)$
-0.001	-1
-0.01	-1
-0.1	1
0.1	1
0.01	1
0.001	1

Draw a graph.



Point is missing.

The Continuity Principles

(p79)

Let c be a number.
$f(x)$ and $g(x)$ be continuous functions.

The following are continuous

1) $h(x) = c \cdot f(x)$
2) $h(x) = f(x) + g(x)$
   $h(x) = f(x) - g(x)$
3) $h(x) = f(x) \cdot g(x)$
4) $h(x) = \dfrac{g(x)}{f(x)}$ is continuous where $f(x) \neq 0$
5) $h(x) = f(g(x))$

"**Quote**" (Euler):
"A function is continuous if you can draw it without lifting your pen."

Not in textbook. Please copy

Fact: The following are continuous where they are defined

- constants
- Polynomials
- $f(x) = \sqrt{x}$ $(x \geq 0)$
- $g(x) = \log(x)$ $(x > 0)$
- $\tan(x)$ $(x \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, e
- $\sin(x)$
- $\cos(x)$
- e^x

Ex:

$$\sin(\sqrt{x} + e^x)$$

is continuous on $x \geq$

Ex: Where is the function

$$f(x) = \frac{x^2+x+1}{x^2-5x+6}$$

continuous?

x^2+x+1 is cont. (poly)
x^2-5x+6 is cont. (poly)

Thus, $f(x)$ is continuous where $x^2-5x+6 \neq 0$.

$$x^2-5x+6 = (x-2)(x-3)$$

Therefore, $f(x)$ is continuous at all $x \neq 2, 3$

Ex: Where is

$$f(x) = \sqrt{x^2-2x}$$

continuous?

We need $x^2-2x \geq 0$
Thus $x(x-2) \geq 0$
Thus, f is continuous on $[0,2]$.

Ex: Where is the function

$$f(x) = \frac{x}{\sqrt{2}\cos(x)-1}$$

continuous?

By the CPs we have:
x is continuous
$\sqrt{2}\cos(x)-1$ is cont.

Thus we need

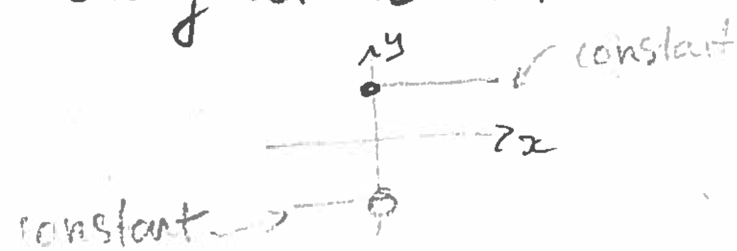
$$\sqrt{2}\cos(x)-1 \neq 0$$
$$\Longleftrightarrow \cos(x) \neq \frac{1}{\sqrt{2}}$$
$$x \neq \frac{\pi}{4}, \frac{-\pi}{4} \pm 2\pi k.$$

Ex: Where is the function

$$g(x) = \frac{x}{|x|}$$

discontinuous?

Only at $x=0$.



Ex: Is the function

$$f(x)=\begin{cases} 2x+1 & x\geq 2 \\ -3x+11 & x<2 \end{cases}$$

continuous at $x=2$?

Recall, we need

$$\lim_{x\to 2} f(x) = f(2) = 5.$$

Determine $\lim_{x\to 2^-} f(x)$

$$\lim_{x\to 2^-} f(x) \overset{*}{=} \lim_{x\to 2^-} 2x+1$$

$$= 2\cdot 2+1 = 5$$

Determine $\lim_{x\to 2^+} f(x)$

$$\lim_{x\to 2^+} f(x) \overset{*}{=} \lim_{x\to 2^+} -3x+11$$

$$= (-3)\cdot 2+11$$

$$= 5$$

Thus, $\lim_{x\to 2} f(x) = 5.$

Ex: For what value k is the function

$$f(x)=\begin{cases} k\sqrt{x} & 0\leq x\leq 10 \\ 20 & x>10 \end{cases}$$

continuous on $x\geq 0$.

We need

$$\lim_{x\to 10^-} f(x) = \underbrace{\lim_{x\to 10^+} f(x)}_{=20}.$$

By CP:

$$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^-} k\sqrt{x}$$

$$= k\sqrt{10}$$

We calculate,

$$20 = k\sqrt{10}$$

$$\Rightarrow k = \frac{20}{\sqrt{10}} = 2\sqrt{10}.$$

Thus, $f(x)$ is continuous if $k = 2\sqrt{10}$.