MAT A29 wk 4a

This week we will cover: §2.5–9

This lecture will cover: §2.5–7

Recall from last week,

AVERAGE RATE OF CHANGE $\quad m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

INSTANT RATE OF CHANGE $\quad m = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{(x+h) - x}$

Ex: Suppose you drive so that $d(t) = 80t$ where $d(t)$ is distance in kilometers and t is time in hours.

- What is your average rate of change?

$$m = \frac{d(x_2) - d(x_1)}{x_2 - x_1} = \frac{80x_2 - 80x_1}{x_2 - x_1}$$

$$= \frac{80(x_2 - x_1)}{x_2 - x_1} = 80.$$

NB: This holds for any x_1 and x_2

- What is your instant rate of change?

$$m = \lim_{h \to 0} \frac{d(x+h) - d(x)}{h}$$

$$= \frac{80(x+h) - 80x}{h} = 80.$$

NB: This holds for any x.

Ex: If $f(x)$ is constant
then what is its average rate of change

$f(x)$ is constant means $f(x) = c$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{c - c}{x_2 - x_1} = 0.$$

Ex: If $f(x)$ is linear
then what is its instant rate of change

$f(x)$ is linear means $f(x) = mx + b$.

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{[m(x+h) + b] - [mx + b]}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = m.$$

Thus, the instant rate of change of a line is its SLOPE.

Ex: Suppose $d(t) = 80t + t^2$

What is your speed at time t?

\# "at time t" indicates instant speed.

$$m(t) = \lim_{h\to 0} \frac{d(t+h) - d(t)}{(t+h) - (t)}$$

$$= \lim_{h\to 0} \frac{[80(t+h) + (t+h)^2] - [80t + t^2]}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{80t} + 80h + \cancel{t^2} + 2th + h^2] - [\cancel{80t} + \cancel{t^2}]}{h}$$

$$= \lim_{h\to 0} \frac{80h + 2th + h^2}{h}$$

$$= \lim_{h\to 0} 80 + 2t + h = 80 + 2t.$$

Therefore, at time t,

you are travelling $(80 + 2t)$ km/h.

Ex: If $d(t) = t^3$ what is the speed at time t?

$$m(t) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{(x+h) - x} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{x^3} + 3x^2h + 3xh^2 + h^3] - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} 3x^2 + h(\cdots) = 3x^2.$$

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Defⁿ: The DERIVATIVE of $y=f(x)$ is the instantaneous rate of change and we write

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x)$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(Addition Rule)

Thm: $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$

The rate of change of a sum is the sum of the rates of change.

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) + \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$$

Thm (Scaling Rule)

$$\frac{d}{dt}\left(c\cdot f(t)\right) = c\cdot\frac{d}{dt}\left(f(t)\right)$$

Multipliers pull out of derivatives.

NB: This fact allows us to work on derivatives in small parts.

We only need to take derivatives of "simple" functions.

Fact (The Power Rule):

If $k \neq 0$ then $\frac{d}{dx}(x^k) = kx^{k-1}$

Ex: An oak tree grows like:

$$\underset{\text{meters}}{h(t)} = \begin{cases} 2.5\sqrt{t} & t \leq 100 \leftarrow \text{years} \\ 25 & t > 100. \end{cases}$$

How fast is it growing at $t = 25$ yrs?

$$\frac{d}{dt}h(t) = \frac{d}{dt}(2.5\sqrt{t})$$

$$= \frac{d}{dt}(2.5t^{\frac{1}{2}})$$

$\frac{d}{dt}(c \cdot f(t)) = c \cdot \frac{d}{dt}(f(t))$

$$= 2.5\left(\frac{d}{dt}t^{\frac{1}{2}}\right)$$

$$= 2.5\left(\frac{1}{2}t^{\frac{1}{2}-1}\right)$$

$$= 2.5\left(\frac{1}{2}t^{-\frac{1}{2}}\right) = \frac{2.5}{2} \cdot \frac{1}{\sqrt{t}}.$$

We get $\frac{2.5}{2}\frac{1}{\sqrt{25}} = \frac{2.5}{2} \cdot \frac{1}{5} = \frac{1}{4}$ m/yr.

Remember to include units!

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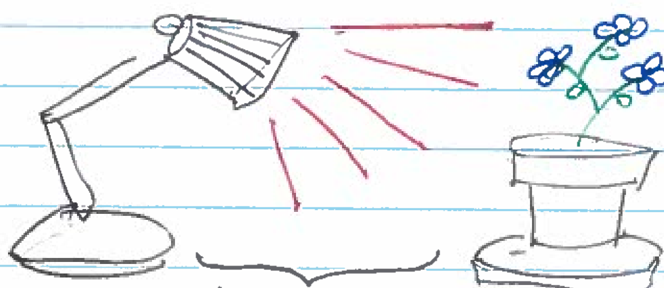
Ex: The amount of energy a plant recieves from a light sorrce is proportional to:

$$e(d) = \frac{1}{d^2}$$

d ← distance between light and plant in meters



d meters

Compare the effect of moving the plant when $d = 1m$ and $d = 10m$.

\# "when $d=1m$" ⟹ instant change.

$$e'(d) = \lim_{h\to 0} \frac{\frac{1}{(d+h)^2} - \frac{1}{d^2}}{(d+h) - d}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(d+h)^2} - \frac{1}{d^2}}{h} = \lim_{h\to 0} \frac{d^2 - (d+h)^2}{h \cdot d^2 (d+h)^2}$$

$$= \lim_{h\to 0} \frac{-2dh - h^2}{h \cdot d^2 (d+h)^2} = \lim_{h\to 0} \frac{-2d-h}{d^2(d+h)^2}$$

$$= -\frac{2}{d^3}$$

$e'(1) = -2 \quad e'(10) = -\frac{1}{500}$

Thus, moving the plant when it is near the light source effects its energy level a lot.

Thm (Quotient Rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

[See pg. 135 for proof]

Ex: If $h(x) = \frac{x^3+1}{x+2}$ what is $h'(x)$ at $x=3$?

Calculate $h'(x)$. Let $f(x) = x^3+1$
$g(x) = x+2$.

$$\frac{d}{dx}h(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Quotient rule

Calculate $f'(x)$ and $g'(x)$

$f'(x) = 3x^2$ Power rule
$g'(x) = 1$

We obtain:

$$h'(x) = \frac{[3x^2](x+2) - [x^3+1](1)}{(x+2)^2}$$

$$= \frac{3x^3+6x^2-x^3-x}{(x+2)^2}$$

$$= \frac{2x^3+6x^2-x}{(x+2)^2}$$

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Ex: Suppose Bob weighs $b(t) = 80 - 5t(t-4)$

Kilograms, time in months

Q1: Is Bob losing/gaining weight during $0 \leq t \leq 1$? $3 \leq t \leq 4$?

Q2: When does Bob start losing weight?

Q1

calculate average for $0 \leq t \leq 1$.

$$m = \frac{80 - (95)}{0 - 1} = 15$$

$+ \Rightarrow$ gain

calculate average for $3 \leq t \leq 4$.

$$m = \frac{95 - 80}{3 - 4} = -15$$

$- \Rightarrow$ loss

when does $b'(t)$ go from + to −

when is $b'(t) = 0$?

Calculate $b'(t)$.

$$b(t) = 80 - 5t(t-4)$$
$$= 80 - 5t^2 + 20t$$

$$b'(t) = -10t + 20$$

$$b'(t) = 0 \Rightarrow t = 2.$$

Thus, Bob gains weight for

$0 \leq t < 2$

and loses weight when

$2 < t \leq 4$.

Worked Examples

Ex: What is the derivative of

$$f(x) = \sqrt{x} + \frac{10 \cdot x}{x^2+1} \, ?$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\sqrt{x} + \frac{10 \cdot x}{x^2+1}\right)$$

$$= \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}\left(\frac{10 \cdot x}{x^2+1}\right) \quad \text{(Sum rule)}$$

$$= \frac{d}{dx}(\sqrt{x}) + 10\,\frac{d}{dx}\left(\frac{x}{x^2+1}\right) \quad \text{(Scaling)}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} + 10\left(\frac{(x^2+1) - x(2x)}{(x^2+1)^2}\right) \quad \text{(Quotient)}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{x}} + \left(\frac{-x^2+1}{(x^2+1)^2}\right)$$

Ex: What is the derivative of $f(x) = x^{\frac{3}{4}} + 2x^3$?

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^{\frac{3}{4}} + 2x^3\right)$$

$$= \frac{d}{dx}\left(x^{\frac{3}{4}}\right) + \frac{d}{dx}(2x^3) \quad \text{(Sum rule)}$$

$$= \frac{3}{4}x^{-\frac{1}{4}} + 6x^2 \quad \text{(Power and Scaling)}$$

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Summary of Derivative Rules

If $f(x)$ and $g(x)$ are differentiable functions.

- $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$
- $\frac{d}{dx}\left(c \cdot f(x)\right) = c \cdot \frac{d}{dx}(f(x))$

If $k \neq 0$ then $\frac{d}{dx}(x^k) = kx^{k-1}$.

Next lecture:
- products
- special functions
- chain rule.