MAT A29 Wk 6a

"Welcome back from Reading Week"
- visits with family
- good meals
- studying

Review of Curve Sketching (p204)

1. Intercepts # $f(x)=0$ and $f(0)=y$
2. Asymptotes # $\lim_{x\to c} f(x)=\infty$ and $\lim_{x\to\infty} f(x)$
3. Derivatives # $f'(x)$ and $f''(x)$.
4. Critical points # $f'(x)=\text{undef}$ OR $f'(x)=0$
5. Increasing/Decreasing, Relative Extrema — # $f'(x)>0$, # $f'(x)<0$ } TABLE
6. Inflection points # $f''(x)=0$ OR $f''(x)=\emptyset$.
7. Concavity # $f''(x)>0$, # $f''(x)<0$.
8. Drawing!

Curve Sketch of $y = x^3 - x$.

Step 1 : Intercepts

y-intercept: $y = f(0) = 0^3 - 0 = 0$

x-intercepts: $f(x) = 0 \Rightarrow x^3 - x = 0$
$\Rightarrow x(x^2-1) = 0$
$\Rightarrow x(x-1)(x+1) = 0$
$\Rightarrow x = 0, 1, -1.$

Step 2 : Asymptotes

vertical: There are no $x = c$ such that $\lim_{x \to c} f(x) = \infty$

horizontal:
- $\lim_{x \to \infty} f(x) = \infty$ and
- $\lim_{x \to -\infty} f(x) = -\infty$

There are no horizontal asymptotes, nor vertical asymptotes.

Curve sketch of $y = x^3 - x$ (con't)

Step ③ : Derivatives

$$f'(x) = \frac{d}{dx}(x^3 - x) = 3x^2 - 1.$$

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}(3x^2 - 1) = 6x$$

Step ④ : Critical points.

$$f'(x) = 0 \implies 3x^2 - 1 = 0$$
$$\implies 3\left(x^2 - \frac{1}{3}\right) = 0$$
$$\implies 3\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) = 0$$
$$\implies x = \frac{1}{\sqrt{3}} \text{ or } x = -\frac{1}{\sqrt{3}}$$

Step ⑤ : Increasing / Decreasing / Rel. Extrema

# critical point	$x = -\frac{1}{\sqrt{3}}$		$x = \frac{1}{\sqrt{3}}$
# Interval	$(-\infty, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, \infty)$
# Test value	$x = -2$	$x = 0$	$x = 2$
# $f'(x)$	$f'(-2) = 3(-2)^2 - 1 = 11$	$f'(0) = 3 \cdot 0^2 - 1 = -1$	$f'(2) = 3 \cdot 2^2 - 1 = 11$
# Inc/Dec	$\nearrow$	$\rightarrow$	$\nearrow$
# Conclusion	$x = -\frac{1}{\sqrt{3}}$ is rel. max		$x = \frac{1}{\sqrt{3}}$ is a rel. min

Curve Sketch of $y = x^3 - x$ (con't)

Step (6): Inflection Points

$$f''(x) = 0 \implies 6x = 0$$

$x = 0$ is a possible inflection point.

Step (7): Concavity

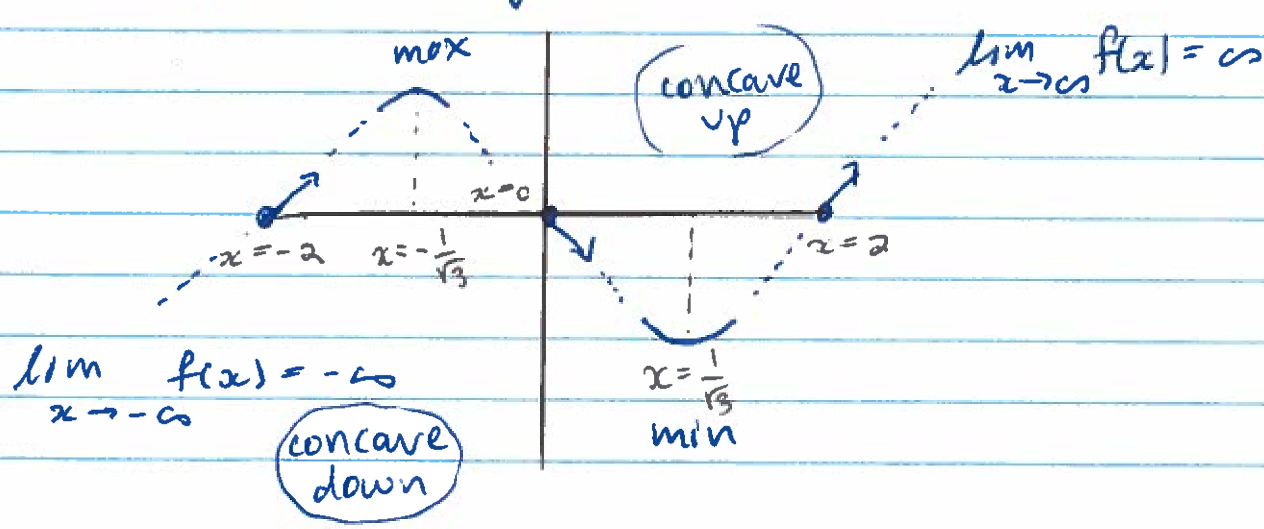
$$f''(x) > 0 \implies 6x > 0$$
$$\implies x > 0$$

If $x > 0$ then $f(x)$ is concave UP.

$$f''(x) < 0 \implies 6x < 0$$
$$\implies x < 0$$

If $x < 0$ then $f(x)$ is concave DOWN

Step (8): Drawing!



Curve Sketch of $y = \frac{1}{x^2 - 2}$

Step ①: Intercepts

y-intercept: $y = f(0) = \frac{1}{0^2 - 2} = -\frac{1}{2}$

x-intercepts: $f(x) = 0 \Rightarrow 0 = \frac{1}{x^2 - 2}$

No solutions.

Step ②: Asymptotes

horizontal: $\lim_{x \to \infty} \frac{1}{x^2 - 2} = 0$

$\lim_{x \to -\infty} \frac{1}{x^2 - 2} = 0$

vertical: $f(x)$ is undefined where $x^2 - 2 = 0$
Thus $x = -\sqrt{2}$ and $x = \sqrt{2}$ will be vertical asymptotes.

$$\begin{cases} \lim_{x \to \sqrt{2}^+} f(x) = \lim_{x \to \sqrt{2}^+} \frac{1}{x^2 - 2} = +\infty \\ \lim_{x \to \sqrt{2}^-} f(x) = \lim_{x \to \sqrt{2}^-} \frac{1}{x^2 - 2} = -\infty \end{cases}$$

x is bigger than $\sqrt{2}$

$$\begin{cases} \lim_{x \to -\sqrt{2}^+} f(x) = -\infty \\ \lim_{x \to -\sqrt{2}^-} f(x) = +\infty \end{cases}$$

absolute value $< \sqrt{2}$

abs. value $> \sqrt{2}$

Curve Sketch of $y = \frac{1}{x^2-2}$

Step ③ : Derivatives

$$f'(x) = \frac{d}{dx}\left[\frac{1}{x^2-2}\right] = \frac{-2x}{(x^2-2)^2}$$

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left[\frac{-2x}{(x^2-2)^2}\right]$$

$$= \frac{\frac{d}{dx}[-2x]\cdot(x^2-2)^2 - (-2x)\cdot\frac{d}{dx}[(x^2-2)^2]}{(x^2-2)^4}$$

"The gears of arithmetic grind slowly..."

$$= \frac{-2(x^2-2)^2 + 2x\cdot 2(x^2-2)\cdot 2x}{(x^2-2)^4}$$

$$= \frac{(x^2-2)[-2(x^2-2) + 2x\cdot 2\cdot 2x]}{(x^2-2)^4}$$

$$= \frac{-2x^2+4+8x^2}{(x^2-2)^3} = \frac{6x^2+4}{(x^2-2)^3}$$

Step ④ : Critical points

$f'(x)$ is undef $\Rightarrow (x^2-2)^2 = 0$

$\Rightarrow x^2 - 2 = 0$

$\Rightarrow x = -\sqrt{2}$ OR $x = \sqrt{2}$

$f'(x) = 0 \Rightarrow x = 0$

Step 5: Increasing/Decreasing/Relative Extrema

	$x=-\sqrt{2}$		$x=0$		$x=\sqrt{2}$	
$(-\infty,-\sqrt{2})$		$(-\sqrt{2},0)$		$(0,\sqrt{2})$		$(\sqrt{2},\infty)$
$x=-2$		$x=-1$		$x=1$		$x=2$
$f'(-2)=1$		$f'(-1)=2$		$f'(1)=-2$		$f'(2)=-1$
$\nearrow$		$\nearrow$		$\searrow$		$\searrow$
	no conclusion		$x=0$ is a rel. max		no conclusion	

$$f'(-2)=\frac{-2(-2)}{((-2)^2-2)^2}=\frac{4}{(4-2)^2}=\frac{4}{2^2}=1$$

$$f'(-1)=\frac{-2(-1)}{((-1)^2-2)^2}=\frac{2}{(1-2)^2}=\frac{2}{(-1)^2}=2$$

Step 6: Inflection Points

$f''(x)$ is undef $\Rightarrow \dfrac{6x^2+4}{(x^2-2)^3}$ is undef

$\Rightarrow x=-\sqrt{2}$ or $x=\sqrt{2}$

$f''(x)=0 \Rightarrow \dfrac{6x^2+4}{(x^2-2)^3}=0$ No solutions

Step 7 : Concavity

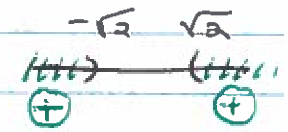
$$f''(x) > 0 \Rightarrow \frac{6x^2+4}{(x^2-2)^3} > 0$$

\# Numerator is always positive.

$$\Rightarrow (x^2-2)^3 > 0$$

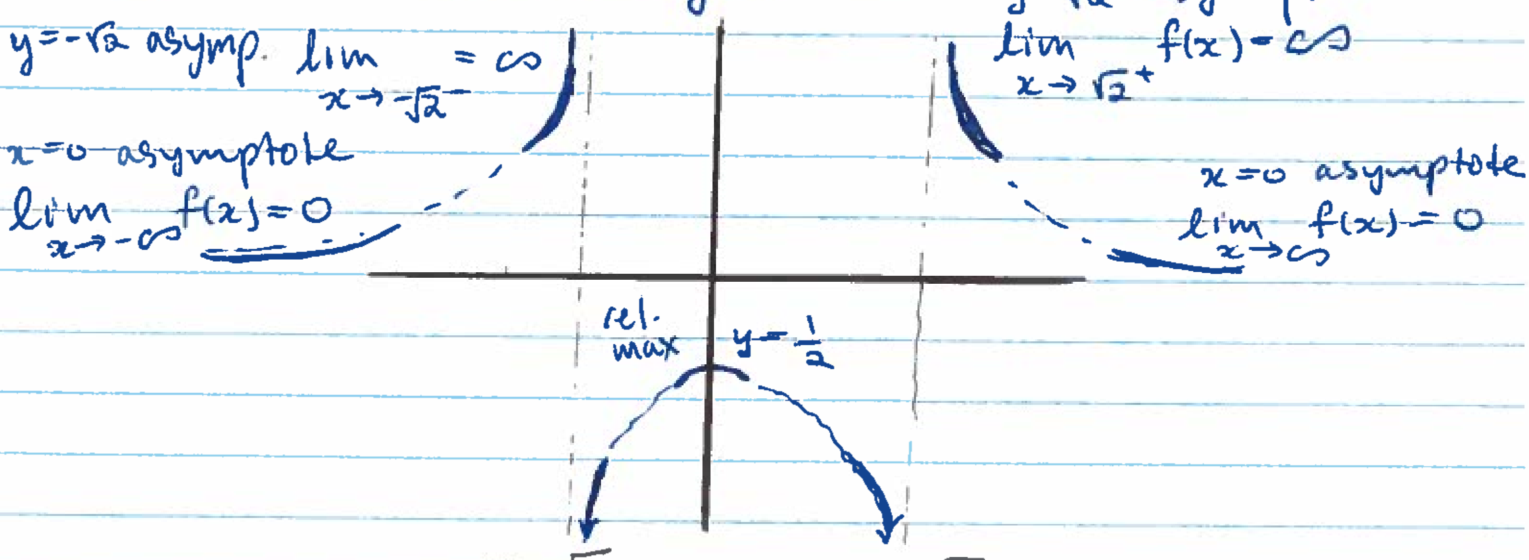
$$\Rightarrow x^2 - 2 > 0$$

$$\Rightarrow x > \sqrt{2} \text{ or } x < -\sqrt{2}$$



$$f''(x) < 0 \Rightarrow -\sqrt{2} < x < \sqrt{2}$$

Step 8 : Drawing!



$$\lim_{x\to -\sqrt{2}^+} f(x) = \lim_{x\to\sqrt{2}^-} f(x) = -\infty$$