Solving Logarithmic Equations

Ex: Solve $\log(10x+5) - \log(x-4) = \log(2)$

$\log(a/b) = \log(a) - \log(b)$

$\log(10x+5) - \log(x-4) = \log(2)$

$\Leftrightarrow \log\left(\frac{10x+5}{x-4}\right) = \log(2)$ # $x=y \Leftrightarrow 10^x = 10^y$

$10^{\log(t)} = t$

$\Leftrightarrow \frac{10x+5}{x-4} = 2$

$\Leftrightarrow 10x+5 = 2(x-4)$

$\Leftrightarrow 8x+13 = 0 \Leftrightarrow x = -\frac{13}{8}$.

Ex: Solve $\log_3 3 + \log_3(x+1) - \log_3(2x-7) = 4$

$\Leftrightarrow 1 + \log_3(x+1) - \log_3(2x-7) = 4$

$\Leftrightarrow \log_3\left(\frac{x+1}{2x-7}\right) = 3 \Leftrightarrow \frac{x+1}{2x-7} = 3^3 = 27$

$\Leftrightarrow x+1 = 27(2x-7)$

$\Leftrightarrow -53x + 170 = 0$

$\Leftrightarrow x = \frac{170}{53}$.

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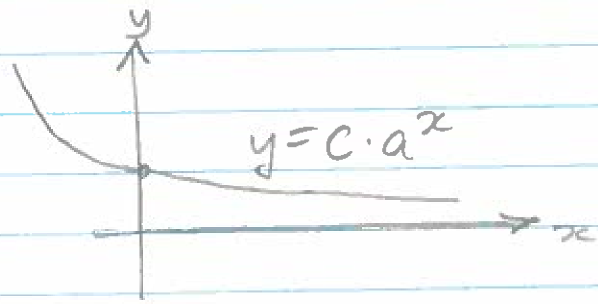
Exponential Decay

Recall, $y = C \cdot a^x$ with $a < 1$ is the model for exponential decay.



Any such equation can be written:

$$y = C \cdot e^{-kx} \quad (C, k > 0)$$

$a < 1 \iff -k = \ln(a) < 0$

Defn: k is the DECAY RATE of $y = Ce^{-kx}$.

Radioactive Decay

Every radioactive substance disintegrates at a rate proportionate to its mass.

Defn: The HALF-LIFE t of a substance is the amount of time necessary for a sample to disintegrate half its mass.

Ex: Find the decay rate of a substance with half life T.

Let $M(t)$ be mass at time t.

$$M(t) = M(0)e^{-kt}$$ # Exponential decay.

$$M(T) = \frac{1}{2}M(0)$$ # Half-life

We now solve for k.

$$\frac{1}{2}M(0) = M(0)e^{-kT}$$

$$\frac{1}{2} = e^{-kT}$$

$$\ln\left(\frac{1}{2}\right) = -kT$$

$$-\ln(2) = -kT$$

$$k = \frac{\ln(2)}{T}$$

Thus, a substance with half-life T has decay rate

$$\frac{\ln(2)}{T}$$

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Radio Carbon Dating.

When an organism eats/breathes it picks up carbon-14 atoms for the environ.

Once an organism dies it stops getting C-14 and the C-14 in it starts to decay.

Fact: The half-life of C-14 is 5750 years.

Ex: Suppose a specimen is found to have (1/64)th the amount of C-14 when it was alive. How old is the fossil?

$$M(t) = M(0) \cdot e^{-kt}$$

$$\frac{1}{64} M(0) = M(0) \cdot e^{-\left(\frac{\ln(2)}{T}\right)t}$$

$$2^{-6} = e^{-\left(\frac{\ln(2)}{T}\right)t}$$

$$-6\ln(2) = -\left(\frac{\ln(2)}{T}\right)t$$

$$t = 6 \cdot T$$

Thus, the fossil is $\sim 6 \cdot 5750$ years old.

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Ex: Find $\frac{dy}{dx}$ if $y = 3^x$

$$y = 3^x = [3]^x = [e^{\ln(3)}]^x$$
$$= e^{[x \cdot \ln(3)]}$$

$$\frac{dy}{dx} = \frac{d e^{x \cdot \ln(3)}}{dx}$$
$$= \frac{d e^{x \cdot \ln(3)}}{d(x \cdot \ln(3))} \cdot \frac{d(x \cdot \ln(3))}{dx}$$
$$= [e^{x \cdot \ln(3)}] \cdot \ln(3) = \ln(3) \cdot 3^x.$$

Ex: Find $\frac{dy}{dx}$ if $y = \log_3 x$.

$$\Longleftrightarrow 3^y = x$$
$$\Longleftrightarrow 3^y \cdot \ln(3) \cdot \frac{dy}{dx} = 1$$
$$\Longleftrightarrow \frac{dy}{dx} = \frac{1}{\ln(3) \cdot 3^y} = \frac{1}{\ln(3) \cdot 3^{\log_3(x)}}$$
$$= \frac{1}{\ln(3) \cdot x}.$$

Fact: In general $y = a^x \Rightarrow \frac{dy}{dx} = \ln(a) \cdot a^x$

$$y = \log_a(x) \Rightarrow \frac{dy}{dx} = \frac{1}{\ln(a) \cdot x}.$$

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This week: § 4.4 Exponential Growth/Decay

§ 4.5 Derivatives of Exp/Log

Semi-Log / Log-Log graphs.

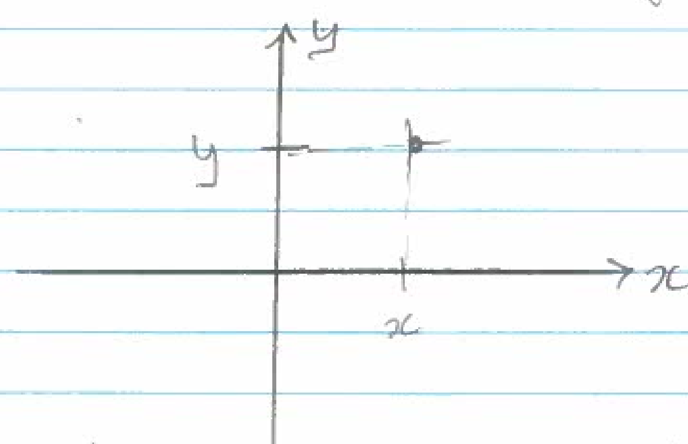
Review of "Standard Graphing"

We give a brief physical review of graphing.

To graph a table of values:

x	$f(x)$
-3	-7
-2	3
0	5
1	2.14
2	6

(1) Draw the Cartesian plane.



(2) Decide on a scale (eg. 1 unit = 1 cm)

(3) Plot all the points.

To plot a point: with ruler

(a) Measure x units along the x-axis.

(b) Measure y units along the y-axis

(c) Label the point (x, y).

Semi-Log Graphing

As before, 1. Draw the Cartesian plane.
2. Decide on a scale (e.g. 10 unit = 5cm)
3. Plot all the points.

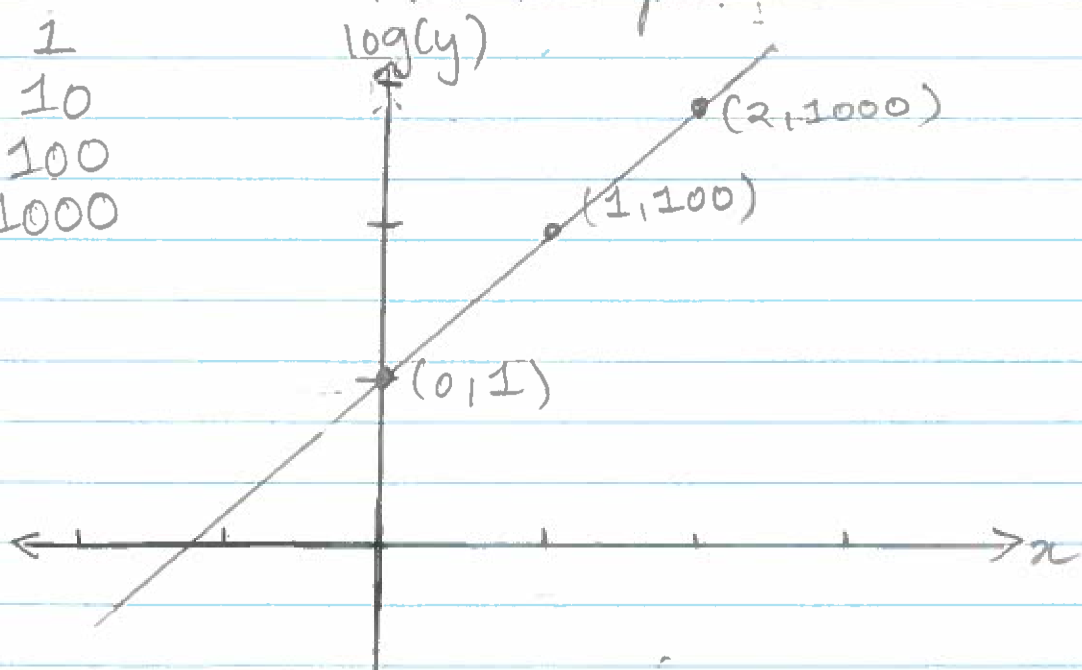
To plot a point: a. Measure x along x-axis.
b. Measure $\log(y)$ units along y-axis
c. Label the point (x, y).

Ex: Draw the semi-log graph of $y = 10^{x+1}$

Make a table of values.

x	y
-2	$1/10$
-1	1
0	10
1	100
2	1000

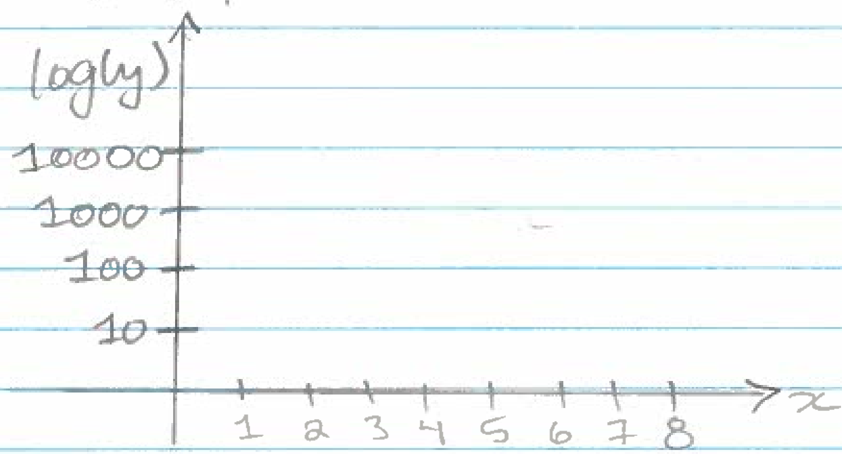
Decide on a scale 2cm = 1 unit.
Plot the points



To plot (1, 100)
- measure (1 unit)(2 cm/unit) = 2cm on x-axis.
- measure ($\log(100)$ units)(2cm/unit) = 4cm on y-axis

Labelling the Axes

In a semi-log plot we label the axes



NB: The numerical difference between adjacent entries on the y-axis increases!

Why do we do this?

- Biological systems exhibit exponential growth
- Cartesian graphs are readable when $x \approx y$.
- $y = C \cdot 10^{kx} \rightsquigarrow \log(y) = \log C + kx$.
- Curve fitting

☆ - The biology department says we have to.

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Log-Log Graphing.

As before
1. Draw the Cartesian plane.
2. Decide on a scale (e.g. 3cm = 1unit)
3. Plot all the points.

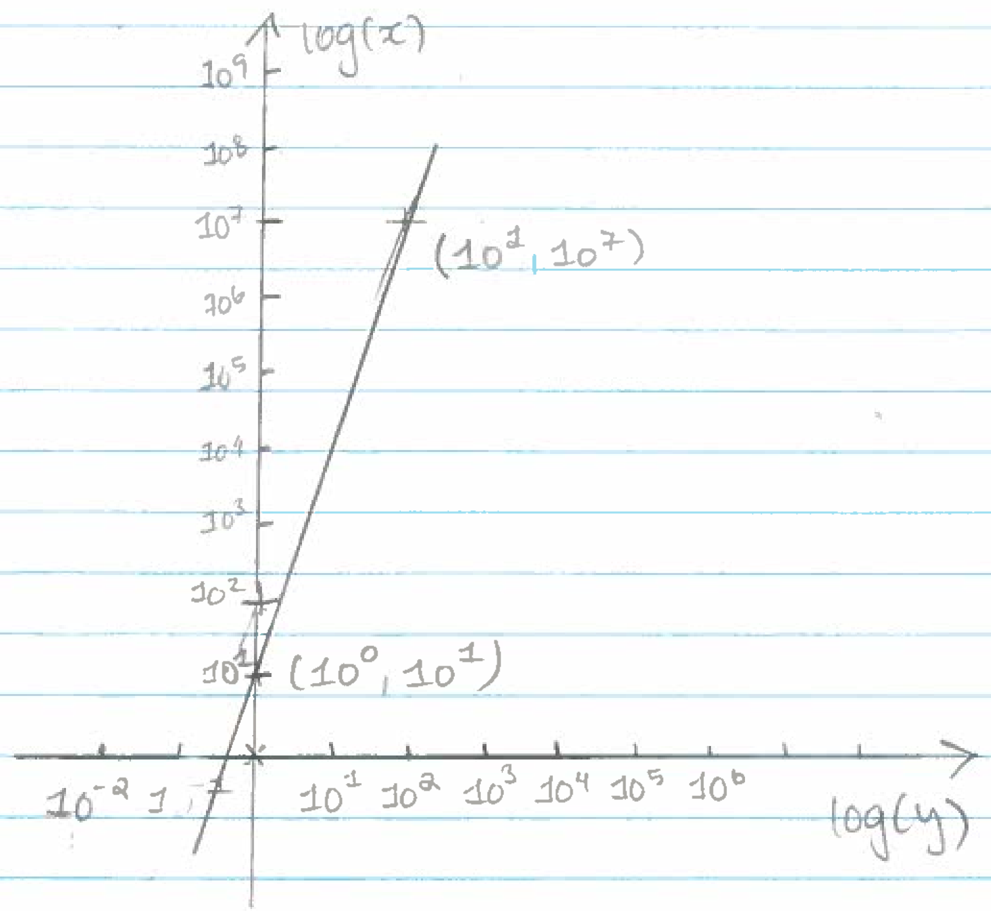
To plot a point:
(a) Measure log(x) units along x-axis.
(b) Measure log(y) units along y-axis.

Ex: Draw the log-log graph of $y = 10x^3$.

Make a table of values. # 1 unit = 1 cm

x	y
10^{-2}	10^{-5}
10^{-1}	10^{-2}
10^{0}	10^{1}
10^{1}	10^{4}
10^{2}	10^{7}
10^{3}	10^{10}

These values are NOT negative.



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Summary:

How to plot a point:

Standard:

measure x units along x axis
y units along y axis

Semi-Log

measure x units along x axis
$\log(y)$ units along y axis
label the point (x, y)!

Log-Log

measure $\log(x)$ units along x axis
$\log(y)$ units along y axis
label the point (x, y)!

Advantages

Semi-Log:

$C\,10^{kx} \rightsquigarrow \log(c) + kx$
appears linear

Log-Log:

$c \cdot x^k \rightsquigarrow \log(c) + k\log(x)$
appears linear