Yay! We made it!

This week: Integration by Parts (§5.6)
volume (§5.8?)
Review on Thursday

Course evaluations are still online.

Integration by Parts

Recall the product rule

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

This gives

$$uv = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx$$

$$= \int u\,dv + \int v\,du$$

$$\Rightarrow \int u\,dv = uv - \int v\,du.$$

Or, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

NB: Differentiate u and Integrate v.

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Ex: Use integration by parts to find $\int x e^x dx$

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv}$$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

$dv = e^x dx \Rightarrow \frac{dv}{dx} = e^x$

$\Rightarrow v = e^x$

$= uv - \int v du$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

Check: $\frac{d}{dx}[xe^x - e^x]$

$= e^x + xe^x - e^x$

$= xe^x$

Ex: $\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv}$

$u = x \Rightarrow du = dx$

$dv = \sin x \, dx$

$\Longrightarrow v = -\cos(x)$ //

$= uv - \int v du$

$= -x\cos(x) + \int -\cos(x) dx$

$= -x\cos(x) + \int \cos(x)$

$= -x\cos(x) + \sin(x) + C.$

Check:

$\frac{d}{dx}[-x\cos(x) + \sin(x)]$

$= -\cos(x) + x\sin(x) + \sin(x)$

$= x\sin(x)$

Tricky Examples

Weird parts

Ex: $\int \underbrace{\ln(x)}_{u}\, \underbrace{dx}_{dv}$

$$\left[\begin{array}{l} u=\ln(x) \Rightarrow du=\frac{1}{x}dx \\ dv=dx \Rightarrow v=x \end{array}\right.$$

$= uv - \int v\, du$

$= \ln(x)\cdot x - \int x\cdot \frac{1}{x}dx = \ln(x)\cdot x - \int 1\, dx$

$= \ln(x)\cdot x - x + C.$

Double

Ex: $\int \underbrace{x^2}_{u}\, \underbrace{e^x dx}_{dv}$

$$\left[\begin{array}{l} u=x^2 \Rightarrow du=2x\,dx \\ dv=e^x dx \Rightarrow v=e^x \end{array}\right.$$

$= uv - \int v\, du$ (Integration by parts!)

$= x^2e^x - 2\int \underbrace{x}_{s}\, \underbrace{e^x dx}_{dt}$

$$\left[\begin{array}{l} s=x \Rightarrow ds=dx \\ dt=e^x dx \Rightarrow t=e^x \end{array}\right.$$

$= x^2e^x - 2\left[st - \int t\, ds\right]$

$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$

$= x^2e^x - 2xe^x + 2e^x + C.$

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3.5

Ex: {Double} $\int e^x \sin(x)\,dx$

$$= \int \underbrace{\sin(x)}_{u} \underbrace{e^x\,dx}_{dv}$$

(!) dv should be easy to integrate.

$$= uv - \int v\,du$$

$u = \sin(x) \Rightarrow du = \cos(x)\,dx$

$dv = e^x\,dx \Rightarrow v = e^x$

$$= \sin(x)e^x - \int e^x \cos(x)\,dx$$

$$= \sin(x)e^x - \int \underbrace{\cos(x)}_{s} \underbrace{e^x\,dx}_{dt}$$

$s = \cos(x) \Rightarrow ds = -\sin(x)\,dx$

$dt = e^x\,dx \Rightarrow t = e^x$

$$= \sin(x)e^x - \left[st - \int t\,ds\right]$$

$$= \sin(x)e^x - \left[e^x\cos(x) + \int \sin(x)e^x\,dx\right]$$

$$= \sin(x)e^x - e^x\cos(x) - \int \sin(x)e^x\,dx$$

Thus,

$$2\int \sin(x)e^x\,dx = \sin(x)e^x - e^x\cos(x)$$

Integrating Rational Functions

We have seen how to integrate certain special rational functions

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Now we can reduce to this example.

Ex: $\int \frac{1}{x^2-1}\,dx$

$A(x+1) + B(x-1) = 1$

$$= \int \frac{1}{(x-1)(x+1)}\,dx \qquad \Rightarrow \begin{cases} A = -B \\ A - B = 1 \end{cases}$$

$$= \int \frac{A}{x-1} + \frac{B}{x+1}\,dx$$

$$= \int \frac{1/2}{x-1} - \frac{1/2}{x+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+c| + C.$$

Steps:
1. Factor denominator
2. Split in to factors
3. Solve for numerators
4. Integrate.

Ex : $\int \frac{1}{x^2-5x+6}\,dx$

$$= \int \frac{1}{(x-3)(x-2)}\,dx$$

$$= \int \frac{A}{x-3} + \frac{B}{x-2}\,dx$$

$A(x-2)+B(x-3)=1$

$$\Rightarrow \begin{cases} A+B=0 \\ -2A-3B=1 \end{cases}$$

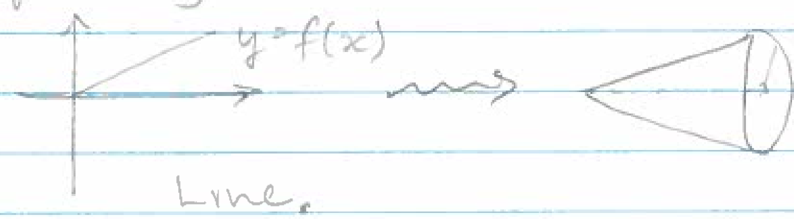
$$\Rightarrow A=1 \quad B=-1$$

$$= \int \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \ln|x-3| - \ln|x-2| + C.$$

Volume by Integration

Def^n: A BODY OF REVOLUTION is a 3D body obtained by taking the graph of a function $y=f(x)$ and spinning it around the x-axis.

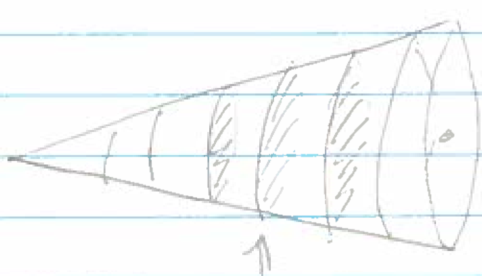


(!) Bodies of revolution always have an axis of rotational symmetry.

Thm: The volume of a body of revolution between $x=a$ and $x=b$ is:

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Pf: Slice body in to disks and use Riemann summation

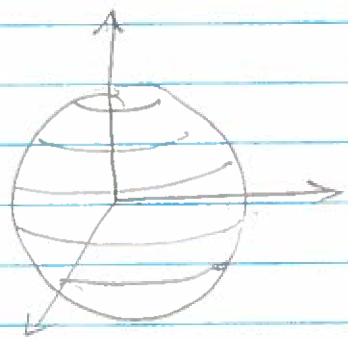


Area $= \pi [f(x)]^2$

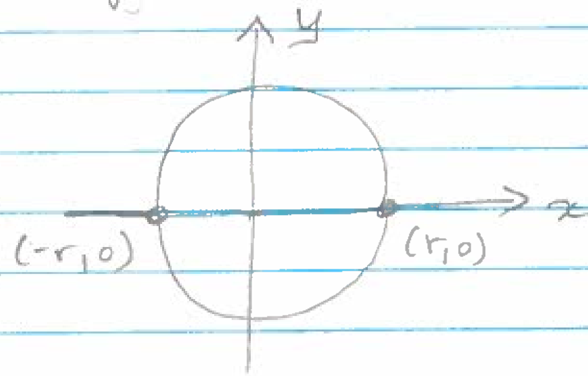
Volume $=$ Area $\cdot \, dx$

Ex: Find the volume of a sphere of radius r.

Draw a diagram



Write the sphere as a body of revolution.



$$f(x) = \sqrt{r^2 - x^2}$$

Thus,

$$V = \int_{-r}^{r} \pi \left[f(x) \right]^2 dx$$

$$= \int_{-r}^{r} \pi \left[\sqrt{r^2 - x^2} \right]^2 dx$$

$$F(x) = xr^2 - \tfrac{1}{3}x^3$$

$$= \int_{-r}^{r} \pi \left[r^2 - x^2 \right] dx$$

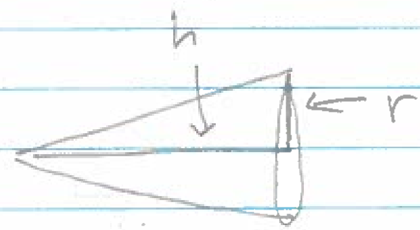
$$F'(x) = r^2 - x^2$$

$$= \pi \left[\left(r^3 - \tfrac{1}{3}r^3 \right) - \left(-r^3 + \tfrac{1}{3}r^3 \right) \right]$$

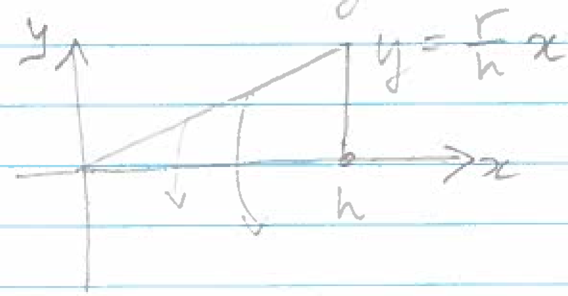
$$= \pi \left[2r^3 - \tfrac{2}{3}r^3 \right] = \frac{4\pi}{3} r^3$$

Ex: Find the volume of a cone of height h and base radius r.

Draw a picture



Write as a body of revolution



$$V = \int_0^h \pi \left[\frac{r}{h} x \right]^2 dx$$

$$= \pi \left(\frac{r}{h} \right)^2 \int_0^h x \, dx$$

$$= \pi \left(\frac{r}{h} \right)^2 \left[\frac{1}{2} h^2 - \frac{1}{2} 0 \right]$$

$$= \frac{\pi r^2}{2h}$$

Cross - Sectional Area

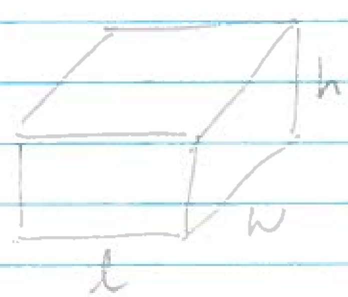
Thm: If the cross-sectional area of an object is $A(x)$ then its volume between $x=a$ and $x=b$ is:

$$V = \int_a^b A(x)\,dx$$

(!) Bodies of revolution are the special case $A(x) = \pi [f(x)]^2$ with circular cross-sections.

Ex: Find the volume of a rectangular prism with dimensions $l \times w \times h$.

Draw a picture

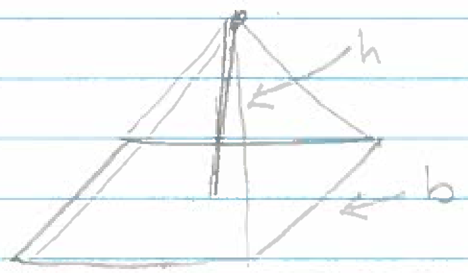


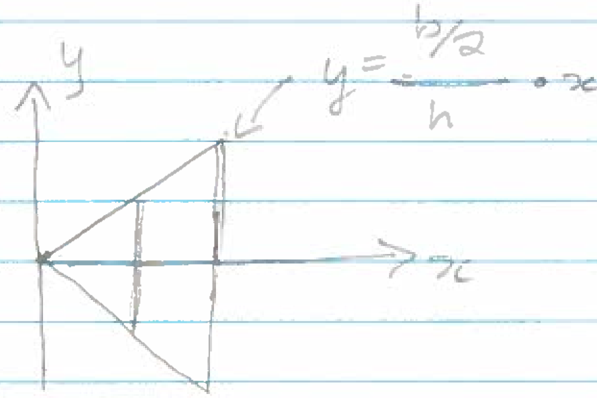
We have $A(x) = wh$

$$V = \int_0^l wh\,dx$$

$$= whl - wh \cdot 0 = whl$$

Ex: Find the volume of a square base pyramid with height h and base b. Assume the tip of the pyramid is above the center of the base.





Thus the cross-sectional area is $\left[\frac{b}{h}x\right]^2$

$$v = \int_0^h \left[\frac{b}{h}x\right]^2$$

$$= \left(\frac{b}{h}\right)^2 \int_0^h x^2 = \left(\frac{b}{h}\right)^2 \cdot \frac{1}{3}h^3$$

$$= \frac{b^2 h}{3}$$

Ex: Assume a cat scan reveals that Parker's cat has cross-section area

$$A(x) = \begin{cases} 10 - x^2 & \text{for } -\sqrt{10} \le x \le 10 \\ 0 & \text{otherwise.} \end{cases}$$