A31 Wk 10a

This week: § 3.1 – 3.3
curve-sketching and Derivatives

"How do the derivatives of $f(x)$ affect the shape of the graph?"

Minima and Maxima

Defn: $x = c$ is a LOCAL MAXIMUM if:
there is $\delta > 0$ such that:
$x \in (c-\delta, c+\delta) \Rightarrow f(x) \leq f(c)$

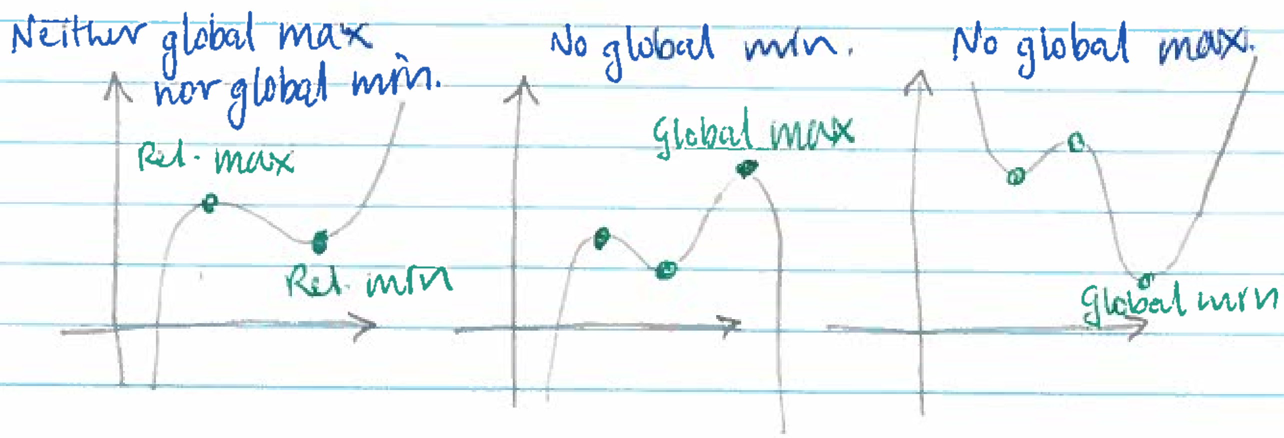
$x = c$ is a LOCAL MINIMUM if:
$x \in (c-\delta, c+\delta) \Rightarrow f(c) \leq f(x)$

The term "local" means "near the value $x = c$"

Defn: $x = c$ is a GLOBAL MAX if:
For all x, $f(x) \leq f(c)$

$x = c$ is a GLOBAL MIN if:
For all x, $f(c) \leq f(x)$

Ex:



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Defn: $x=c$ is a CRITICAL POINT of $f(x)$ if:

Either $f'(c)=0$ or
$f'(c)$ does not exist.

Ex: Find the critical point of $f(x)=|x|$.

Write out derivative

$$f(x)=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases} \Longrightarrow f'(x)=\begin{cases} 1 & x>0 \\ \text{undef} & x=0 \\ -1 & x<0 \end{cases}$$

Thus, the only critical point is $x=0$.

Ex: Find the critical points of $f(x)=x^3-x$.

Find the derivative

$$f'(x)=3x^2-1$$

$$f'(x)=0 \Longrightarrow x=-\frac{1}{\sqrt{3}} \text{ OR } x=\frac{1}{\sqrt{3}}.$$

Thus, the only critical points are $x=-\frac{1}{\sqrt{3}}$ and $x=\frac{1}{\sqrt{3}}$.

Ex: Find the critical points for $f(x)=\frac{1}{1-x^2}$.

Differentiate

$$f'(x)=-\frac{1}{(1-x^2)^2}\cdot(-2x) \Longrightarrow x=\pm 1$$

"Why are critical points important?"

Thm (Fermat):

If $x=c$ is a local extrema
then $x=c$ is a critical point.

Pf (for local max) (!) Reverse is false.

Suppose $x=c$ is a local max.

If $f'(c)$ does not exist then we are done.

Suppose $f'(c)$ does exist. We show $f'(c)=0$

By maximality:
There is $\delta>0$ so that
$x\in(c-\delta, c+\delta) \implies f(x)\leq f(c)$

For $x\in(c, c+\delta)$ we have:
$x-c>0$ and
$f(x)-f(c)\leq 0$.

Thus,
$$\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\leq 0.$$

For $x\in(c-\delta, c)$ we have:
$x-c<0$ and so,

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\geq 0.$$

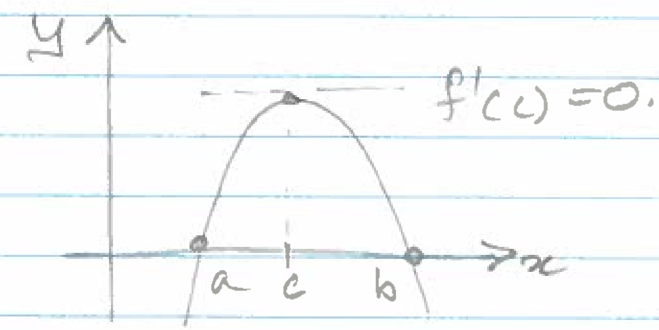
$\implies f'(c)=0.$

Thm (Rolle) If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and

$$f(a) = f(b) = 0$$

then there is $c \in (a,b)$ so that $f'(c) = 0$.

Ex:



Pf: Every continuous function on a closed interval achieves an absolute max and absolute min. $x=c$. *proof delayed*

By Fermat, this is a critical point.

Since $f(x)$ is differentiable $f'(c)$ exists.

Again, critical point $\Rightarrow f'(c) = 0$.

The Closed Interval Method

Thm (Extreme Value)

If $f(x)$ is continuous on a closed interval $[a,b]$ then there are $m, M \in [a,b]$ such that:

$f(m)$ is an absolute min

proof delayed $f(M)$ is an absolute max.

(!) We might have m or M an endpoint.

To find the absolute max/min:

(1) Find critical points in interval.

(2) Check function at critical points

(3) check function at end points.

Ex: Find the max of $y=x^2$ on $[1,2]$.

critical points

$f'(x) = 2x = 0$ $\quad$ $f(1) = 1$

$\Rightarrow x = 0.$ $\quad$ $f(2) = 4$

$0 \notin [1,2]$. $\quad$ Thus $f(2) = 4$ is abs. max

Ex: Find the abs. max and min of $f(x) = x^2 - 2x + 1$ on $[-2, 2]$. $= (x-1)^2$

Find critical points.

$f'(x) = 2x - 2$

$\Rightarrow x = 1$

compare values

$f(-2) = 5$ Thus, $f(-2)$ is absolute max.

$f(1) = 0$ $f(1)$ is absolute min.

$f(2) = 1$

Ex: Find the absolute max and min of $f(x) = |x|$ on $[-1, 2]$.

Find critical points

$x = 0$

compare values

$f(-1) = 1$ Thus, $f(2)$ is absolute max

$f(0) = 0$ $f(0)$ is absolute min.

$f(2) = 2$

Derivatives and Shape

Recall, Rolle's Theorem:

"If $f(a) = f(b)$ then there is a horizontal tangent in between"

We will need the following generalization:

Thm (Mean Value or MVT):

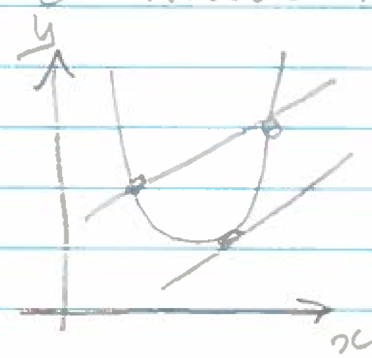
If f is continuous on $[a,b]$ and diff'ble on (a,b) then there is $c \in (a,b)$ such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Pf: (via Rolle's Theorem)

Secant line $\rightarrow$ $$\ell(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

$$g(x) = f(x) - \ell(x)$$ $\leftarrow$ Diff'able and continuous.



Note: $g(a) = f(a) - \ell(a) = 0$
$g(b) = f(b) - \ell(b) = 0$

Thus, $g'(c) = 0$ for some c.

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$\Longrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

QED.

Ex: MVT $\Rightarrow$ Rolle

If $f(a)=f(b)=0$ then

$$f'(c) = \frac{f(a)-f(b)}{b-a} = \frac{0-0}{b-a} = 0.$$

thus Rolle's conclusion holds.

Ex: If you walk 2 km in 1 hr then

there is a moment when you are travelling 2 km/hr.

Pf: Let $d(t)$ be distance travelled up to time t.

We have $d(0)=0$ and $d(1)=2$.

Thus there is $c \in (0,2)$ so that:

$$d'(c) = \frac{d(1)-d(0)}{1-0} = \frac{2-0}{1} = 2.$$

Thus, at some point you travel 2 km/h.

Recall,

Defn: $f(x)$ is INCREASING if:
$a<b \Rightarrow f(a)<f(b)$

Thm: If $f(x)$ is continuous and differentiable and $x\in(a,b) \Rightarrow f'(x)>0$
then $f(x)$ is increasing on (a,b).

Pf (via MVT)

Pick $s,t \in (a,b)$ we have:
There is $c\in(s,t)$ so that

$$\frac{f(t)-f(s)}{t-s} = f'(c) > 0$$

$$\Rightarrow f(t)-f(s) = \underset{+}{f'(c)}\cdot\underset{+}{(t-s)} > 0$$

$$\Rightarrow f(t) > f(s).$$

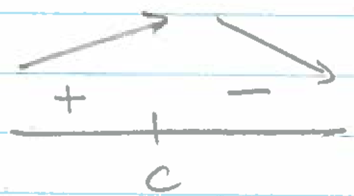
QED.

Exercise: If $f'(x)<0$ then decreasing.

The First Derivative Test

If $c \in (a,b)$ is a critical point of $f(x)$ and (a,b) does not contain any other critical points then:

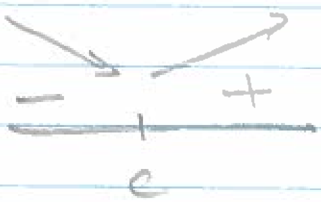
$\boxed{\text{I}}$



$$\left[\begin{array}{l} x \in (a,c) \Rightarrow f'(x) > 0 \\ x \in (c,b) \Rightarrow f'(x) < 0 \end{array}\right.$$

$f(c)$ is a max

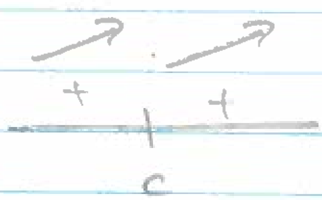
$\boxed{\text{II}}$



$$\left[\begin{array}{l} x \in (a,c) \Rightarrow f'(x) < 0 \\ x \in (c,b) \Rightarrow f'(x) > 0 \end{array}\right.$$

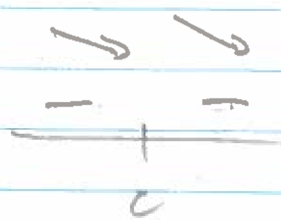
$f(c)$ is a min

$\boxed{\text{III}}$



$f(c)$ is not a local extrema

$\boxed{\text{IV}}$



$f(c)$ is not a local extrema.

represents critical point