Yay! We made it!

This week: L'Hopital's Rule (§3.6)

Anti-Differentiation (§4.4)

Review

Recall,

the expressions "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$"

are not well defined real numbers.

Ex: $\lim\limits_{x \to 0} \frac{7x}{x} = 7$ and $\lim\limits_{x \to 0} \frac{\pi x}{x} = \pi$

Thus, "$\frac{0}{0}$" $= 7$ and "$\frac{0}{0}$" $= \pi$

Ex: $\lim\limits_{x \to \infty} \frac{100x}{x} = 100$ and $\lim\limits_{x \to \infty} \frac{e \cdot x}{x} = e$.

Thus, "$\frac{\infty}{\infty}$" $= 100$ and "$\frac{\infty}{\infty}$" $= e$.

"How do we make sense of these?"

Recall,

linearization is the process where we approximate a function by its tangent line.

Thm (L'Hopital's Rule)

Suppose f and g are differentiable on a punctured interval about $x=c$. and $g(x) \neq 0$ on this interval.

If $\lim_{x \to c} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"}$ or $\text{"}\frac{\infty}{\infty}\text{"}$

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

(The conclusion holds for $c = \infty$ too if f and g are diffable on (N, ∞).)

Pf (case $\frac{0}{0}$)

$$\frac{f'(c)}{g'(c)} = \frac{\lim_{x \to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x \to c} \frac{g(x)-g(c)}{x-c}}$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x \to c} \frac{f(x)}{g(x)}$$

Geometric Intuition

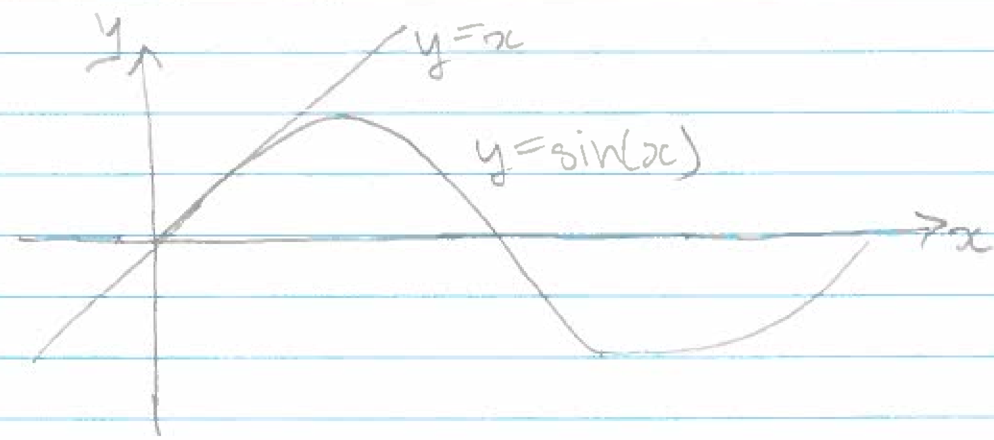
We replace $f(x)$ and $g(x)$ by their tangent lines.

Ex: $\lim_{x \to 0} \frac{\sin(x)}{x}$

The tangent line to $y=\sin(x)$ at $x=0$ has slope $m=\cos(0)=1$ and passes through $(0,0)$.

We get

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$



Ex $\lim_{x\to 0} \frac{x^2}{2^x - 1}$

We see $\lim_{x\to 0} \frac{x^2}{2^x-1} = \frac{\text{"}0\text{"}}{\text{"}0\text{"}}$ and both diff'able.

We get: ← This means we apply L'Hôpital.

$$\lim_{x\to 0} \frac{x^2}{2^x-1} \overset{L'H}{=} \lim_{x\to 0} \frac{2x}{\ln(2)\cdot 2^x} = \frac{0}{1} = 0$$

Ex: $\lim_{x\to 0} \frac{\sin(x)}{x^2} = \frac{\text{"}0\text{"}}{0}$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)}{2x} = \frac{1}{0} \leftarrow \text{DNE}$$

Thus, $\lim_{x\to 0} \frac{\sin(x)}{x^2}$ does not exist.

Ex: $\lim_{x\to \infty} \frac{2x^2+x+7}{3x^2+5x+2} = \frac{\text{"}\infty\text{"}}{\infty}$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{4x+1}{6x+5} = \frac{\text{"}\infty\text{"}}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{4}{6} = \frac{4}{6} = \frac{2}{3}$$

Re-writing Limits

Ex: $\lim_{x \to \infty} x^3 e^{-3x} = \text{“}0 \cdot \infty\text{”}$

$= \lim_{x \to \infty} \frac{x^3}{e^{3x}} = \text{“}\frac{\infty}{\infty}\text{”}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{3x^2}{3e^{3x}} = \text{“}\frac{\infty}{\infty}\text{”}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{6x}{9e^{3x}} = \text{“}\frac{\infty}{\infty}\text{”}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{6}{27e^{3x}} = 0.$

Ex: $\lim_{x \to 0^+} x \ln(x) = \text{“}0 \cdot (-\infty)\text{”}$

$= -\lim_{x \to 0^+} \frac{-\ln(x)}{1/x} = \text{“}\frac{\infty}{\infty}\text{”}$

$\overset{L'H}{=} -\lim_{x \to 0^+} \frac{-1/x}{1/x^2} = -\lim_{x \to 0^+} \frac{-x}{1} = 0.$

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Logs and Limits

Recall,

if $f(x)$ is continuous:

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

Ex: $\lim_{x \to \infty} x^{\frac{1}{x}}$

$$L = \lim_{x \to \infty} x^{\frac{1}{x}}$$

$$\ln(L) = \ln\left(\lim_{x \to \infty} x^{\frac{1}{x}}\right)$$

$$= \lim_{x \to \infty} \ln\left(x^{\frac{1}{x}}\right)$$

$$= \lim_{x \to \infty} \frac{1}{x} \ln(x)$$

$$= \lim_{x \to \infty} \frac{\ln(x)}{x} = \text{“}\frac{\infty}{\infty}\text{”}$$

$$\overset{L'H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

$$\ln(L) = 0 \Longrightarrow L = 1.$$

Ex: $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$$L = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\ln(L) = \ln\left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right)$$

$$= \lim_{n\to\infty} \ln\left[\left(1+\frac{1}{n}\right)^n\right]$$

$$= \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) = \text{"}\infty \cdot 0\text{"}$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \cancel{\frac{-1}{n^2}}}{\cancel{-\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

$$\ln(L) = 1 \implies L = e$$

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Anti-Differentiation

Recall,

$F(x)$ is an ANTI-DERIVATIVE of $f(x)$
if: $F'(x) = f(x)$.

Thm: If $F'(x) = g'(x) = f(x)$ then there is C so that:

$$F(x) = g(x) + C.$$

Pf: $F'(x) - g'(x) = 0$

$\Rightarrow \frac{d}{dx}[F(x) - g(x)] = 0$

$\Rightarrow F(x) - g(x)$ is constant

$\Rightarrow F(x) = g(x) + C.$

Notation:

$$\underbrace{\int \underbrace{f(x)}_{\text{integrand}} dx}_{\text{indefinite integral}} = \underbrace{F(x)}_{\text{anti-derivative}} + \underbrace{C}_{\text{constant of integration}}$$

This is the notation we use for anti-derivatives.

<u>Thm</u>: $\int af(x) + bg(x)\,dx = a\int f(x)dx + b\int g(x)dx.$

<u>Pf</u>: Suppose $F'(x) = f(x)$ and $G'(x) = g(x)$.

\# want to show:
$aF(x) + bG(x)$ is an anti-derivative of
$af(x) + bg(x)$.

$$\frac{d}{dx}\left[aF(x) + bG(x)\right]$$
$$= aF'(x) + bG'(x)$$
$$= af(x) + bg(x)$$

<u>Thm</u>: If $k \neq -1$ then $\int x^k dx = \frac{1}{k+1} x^{k+1} + C.$

<u>Pf</u>: $\frac{d}{dx}\left[\frac{1}{k+1} x^{k+1}\right]$
$$= \frac{1}{k+1} \cdot (k+1) x^{k+1-1}$$
$$= x^k.$$

<u>Thm</u>: $\int \frac{1}{x} dx = \ln|x| + C$

<u>Ex</u>: $\int x^3 + 2x + 1\,dx = \frac{1}{4}x^4 + x^2 + x + C.$

Differentials

"What does dx mean?"

dx means a "small" (infintesimal) change in x.

$\frac{dy}{dx}$ is the relation between:

a small change in y
and a small change in x.

Ex: Suppose $y = x^2$.
Calculate $\frac{\Delta y}{\Delta x}$ for $\Delta x = 0.1$ at $x = 1$

Calculate $\frac{dy}{dx}$ at $x = 0$.

$$\Delta x = (1.1) - (1.0) = 0.1$$
$$\Delta y = (1.1)^2 - (1.0)^2 = 1.21 - 1.0 = 0.21$$

$$\Longrightarrow \frac{\Delta y}{\Delta x} = \frac{0.21}{0.1} = 2.1$$

$$\frac{dy}{dx} = 2x \Longrightarrow \frac{dy}{dx} = 2.$$

$$dx = \lim_{\Delta \to 0} \Delta x$$

Thm: $d\,f(x) = f'(x)\,dx$.

Pf: $\frac{d\,f(x)}{dx} = f'(x) \Longrightarrow d\,f(x) = f'(x)\,dx$

Anti-Diff by Substitution

The chain rule says:

$$\frac{d}{dx} f(y) = f'(y)\frac{dy}{dx}$$

We get:

$$\int f'(y)\frac{dy}{dx} \cdot dx = f(y) + C.$$

Ex: $\int 2x e^{x^2} dx$

$$= \int \frac{dy}{dx} e^{y} dx \qquad \left[\begin{array}{l} y = x^2 \\ \frac{dy}{dx} = 2x \end{array}\right.$$

$$= \int e^{y} dy$$

$$= e^{x^2} + C$$

Check:

$$\frac{d}{dx}\left[e^{x^2}\right] = 2x e^{x}$$

Ex: $\int \frac{1}{2\sqrt{x}} \cos(\sqrt{x})\, dx$

$$\left[\begin{array}{l} y = \sqrt{x} \\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \end{array}\right.$$

$$= \int \frac{dy}{dx} \cos(y)\, dx$$

$$= \int \cos(y)\, dy$$

$$= \sin(y) + C = \sin(\sqrt{x}) + C.$$

Summary

→ $\int f(x)\,dx = F(x) + C$ means

"the anti-derivative of $f(x)$ is $F(x)$"

→ Anti-Differentiation is linear

$$\int a f(x) + b g(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

→ $$\int x^k\,dx = \begin{cases} \frac{1}{k+1} x^{k+1} & k \neq 1 \\ \ln\|x\| & k = -1 \end{cases}$$

→ $$\int \frac{dy}{dx} f'(y)\,dx = \int f'(y)\,dy = f(y) + C.$$

Handy Formulae

$$\int e^x\,dx = e^x + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$