MAT A31 Wk 2a

Exponential functions

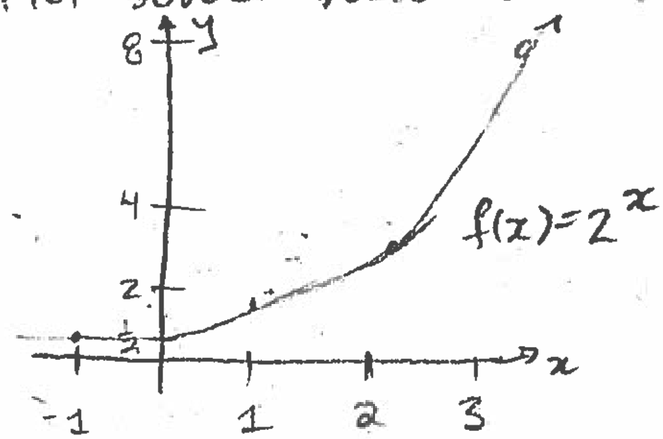
Defn:

$f(x) = A\,b^x \quad (A \neq 0,\ b \neq 1)$

is an EXPONENTIAL function

Ex: Sketch $f(x) = 2^x$

Plot some values



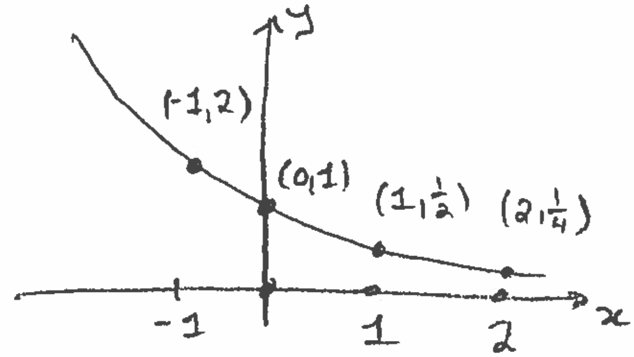
connect the dots.

Remark: We understand

$2^1,\ 2^2,\ 2^3, \ldots$

What does 2^π mean?

Ex: Sketch $f(x) = \left(\frac{1}{2}\right)^x$



Fact:

$b > 1 \leftrightsquigarrow$ exponential growth

$b < 1 \leftrightsquigarrow$ exponential decay

Fact. For any base $b \neq 0$,

- $b^a b^c = b^{a+c}$
- $\dfrac{b^a}{b^c} = b^{a-c}$
- $(b^a)^c = b^{(ac)}$
- $b^0 = 1$

MAT A31 WK 2a

Exponents and Logs

We note that

$$f(x) = b^x$$

is one-to-one and thus has an inverse

$$g(x) = \log_b(x)$$

Fact:

$$\log_b(b^x) = b^{\log_b(x)} = x$$

Ex: $\log_2(2^3) = 3$

$\log_3(9) = 2$

Nota: $\log(x) = \log_{10}(x)$

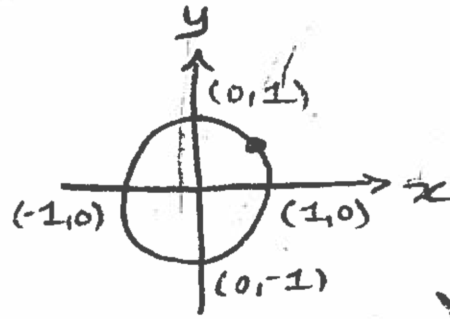
$\ln(x) = \log_e(x)$

"$e \simeq 2.71\ldots$"

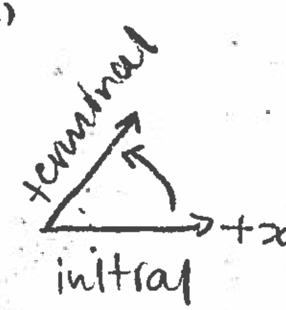
Fact:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x^y) = y\log_b(x)$

The Unit Circle

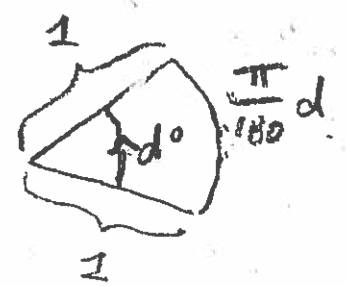


Defn: ANGLE

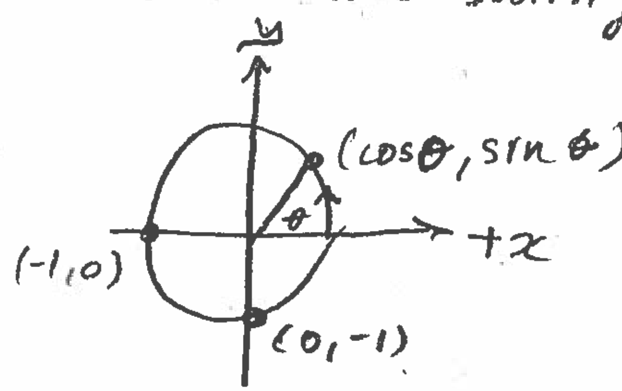


RADIANS

DEGREES



Defn: The functions $\cos\theta$ and $\sin\theta$ satisfy



$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

MAT A31 Wk 2a

Trig Identities

P52

Def^n: An IDENTITY is an equality that needs proof.

Ex (Pythagoras)

$$\sin^2\theta + \cos^2\theta = 1.$$

Pf:

$(\cos\theta, \sin\theta)$ is on unit circle

We know

$$d((0,0), (\cos\theta, \sin\theta)) = 1$$

Thus,

$$\sqrt{(\cos\theta - 0)^2 + (\sin\theta - 0)^2} = 1$$

Therefore

$$\cos^2\theta + \sin^2\theta = 1.$$

QED.

Def^n:

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

Ex: $1 + \tan^2\theta = \sec^2\theta.$

Pf: # write out definitions

$$LHS = 1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta}$$

$$RHS = \sec^2\theta = \frac{1}{\cos^2\theta}.$$

We show, LHS = RHS.

$$LHS = 1 + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

QED

Quantifiers

Def^n:

- FOR ALL : $\forall_{x \in S} P(x)$
  For all $x \in S$, $P(x)$ is true.

- EXISTS : $\exists_{y \in S} P(y)$
  There is $y \in S$, such that $P(y)$ is true.

Ex: Are these true?

Let P be
"$\forall_{x \in \mathbb{R}}\ x+1 > x$"

Let Q be
"$\forall_{x \in \mathbb{R}}\ x^2 > x$"

Let R be
"$\exists_{x \in \mathbb{R}}\ x < x^2$"

Ex (con't)

P is true:

Pf: $1 > 0$ and so

$$1 + x > 0 + x = x$$

QED

Q is false:

There is no way to prove it because
$$\left(\tfrac{1}{2}\right)^2 < \left(\tfrac{1}{2}\right)$$

R is true:

There are lots of examples
$$\left(\tfrac{1}{10}\right)^2 < \tfrac{1}{10}$$

Discuss:

Let $P =$ "$\forall_{x \in \mathbb{R}} \exists_{y \in \mathbb{R}} (x + y = 0)$"

$Q =$ "$\exists_{y \in \mathbb{R}} \forall_{x \in \mathbb{R}} (x + y = 0)$"

MAT A31 Wk 2b

Math and Logic

In mathematics statements may be true or false.

The purpose of a proof of proposition P is to convincingly show:

P is true

P is false.

Ex: Are these true or false?

Let P be "$1+1=2$"

Q be "$5<10$"

∴ It is clear that

P and Q are true.

Ex. Are these true or false?

let S be

"If $x \in \mathbb{R}$ then $x^2 \geq 0$"

Let T be

"If $x \in (1,3)$ then $x \in (\frac{1}{2}, 2)$"

Ex (con't):

A: S is true.

T is false!

If $x = 2.1$ then

$x \in (1,3)$ and

$x \notin (\frac{1}{2}, 2)$.

(Very Important) Fact:

The implication

$P \Longrightarrow Q$

is "true" if:

Q is true

whenever

P is true

Alternatively

"P true" implies "Q true"

Counter-Examples

Def^n:

"A COUNTER-EXAMPLE is an example which shows a statement is false.

Ex: Let P be

"If n is even then n+1 is even"

n=2 is a counterexample to the statement P because

2 is even

2+1 is odd.

Ex: Let S be

"$x \in (7,11) \implies x \in (5,13)$"

Let T be

"$y \in (7,11) \Rightarrow y \in (6,10)$."

which statement has any counterexamples?

Ex: Let Q be

$\forall_{n \in \mathbb{N}} \exists_{k \in \mathbb{N}} (2^k > n)$

Let R be

$\exists_{n \in \mathbb{N}} \forall_{k \in \mathbb{N}} (2^k < n)$

which of these statements has any counterexamples.

In general,

A counterexample to

$P(x) \implies Q(x)$

is x such that

$P(x)$ is true

$Q(x)$ is false.

A counterexample to a quantified statement is a family of examples

Proof as Essay

Proofs have three parts

INTRODUCTION:
- name variables
- suppose hypothesis

BODY:
- do algebra
- argue stuff

CONCLUSION:
- state the result
- write QED.

Ex: Prove
"If n is even then n+1 is odd"

Pf: Suppose n is even. We write $n = 2k, k \in \mathbb{Z}$.
Now consider
$$n + 1 = 2k + 1.$$
The right hand side is odd by definition.
Thus, if n is even then n+1 odd. QED

Ex: Prove
"$\forall_{n \in \mathbb{N}} \exists_{k \in \mathbb{N}} 2^k > n$"

Pf: Suppose n is a natural. We want to find k such that $2^k > n$.

$2^{\log_2(n)} = n$:)

Let k be a natural
$$k > \log_2 n.$$
We get
$$2^k > 2^{\log_2(n)} = n.$$

Thus,
$$\forall_{n \in \mathbb{N}} \exists_{k \in \mathbb{N}} 2^k > n$$
QED.

<u>Proofs with Quantifiers</u> (The Recipe!)

To prove

$\forall x \exists y P(x,y)$

you need to show:

For every x
There is some y
such that
$P(x,y)$ is true.

To prove

$\exists x \forall y P(x,y)$

you need to show:

There is an x such that
For every y
$P(x,y)$ is true.

<u>Ex</u>: How would you prove "Everybody has a soulmate."?

Show that:

For each person x
there is a person y
such that
y is x's soulmate.

This is very difficult!

<u>Ex</u>: How would you prove "There is a real number less than every natural"?

Show there is x such that for each natural n
$x < n$.

e.g. Take $x = -10$.

MAT A31 Wk 2 c

Proofs with Inequalities

<u>Ex</u>: Prove

"$\forall \varepsilon>0 \ \exists n\in\mathbb{N} \ \frac{1}{n}<\varepsilon$"

<u>Ideas</u>:

Examine some cases

$\frac{1}{1}=1 \quad \frac{1}{10}=0.1 \quad \frac{1}{100}=0.01$

Examine the inequality

$\frac{1}{n}<\varepsilon$.

<u>Pf</u>:

For $\varepsilon>0$ we pick $n\in\mathbb{N}$ such that $n>\frac{1}{\varepsilon}$.

We obtain

$$n>\frac{1}{\varepsilon} \Longleftrightarrow \frac{1}{n}<\varepsilon$$

Thus,

for all $\varepsilon>0$ there is $n\in\mathbb{N}$ such that $\frac{1}{n}<\varepsilon$.

QED

<u>Ex</u>: Prove

"If $|x-1|<\frac{1}{10}$ then $|x^2-1|<10$."

<u>Ideas</u>:

Draw the interval

$|x-1|<\frac{1}{10}$

Factor the polynomial

$x^2-1=(x+1)(x-1)$

<u>Pf</u>:

Suppose $|x-1|<\frac{1}{10}$.

We get: # Try to get $x+1$

$$-\frac{1}{10}<x-1<\frac{1}{10}$$

$$2-\frac{1}{10}<x+1<2+\frac{1}{10}$$

$$1<x+1<3$$

Thus, $|x+1|<3$

It follows,

$$|x^2-1|<3\cdot\frac{1}{10}<10.$$

QED