The δ - ε Story

Alice and Bob are students in MAT A31. They are talking in the Math Aid Center.

<u>Bob</u>: Hey, Alice! I proved $\lim_{x\to 2} 2x+3 = 100$.

<u>Alice</u>: I am not so sure. In fact, I can make sure "$\lim_{x\to 2} 2x+3 < 19$."

Take $\varepsilon = 10$.

If we pick $\delta = 5$ we get:

$$0 < |x-3| < \delta = 5$$
$$\Rightarrow 0 < 2|x-3| < 2\cdot 5$$
$$\Rightarrow 0 < |2x-6| < 10$$
$$\Rightarrow 0 < |(2x+3)-9| < 10$$

$$\Rightarrow -10 < (2x+3)-9 < 10$$

$$\Rightarrow -10 < (2x+3) < 19.$$

Thus, we get "$\lim_{x\to 2} 2x+3 < 19$"

<u>Bob</u>: uhh-ohh. Let me check my work.

The δ - ε Story

Bob: okay, Alice. You were right.

I *can* prove $\lim_{x \to 2} (2x+3) = 10$.

Alice: I am still not sure.

In fact, I can prove "$\lim_{x \to 2} 2x+3 < 9 + \frac{1}{2}$".

If we set $\varepsilon = \frac{1}{2}$ then $\delta = \frac{1}{4}$ gives:

$$0 < |x-3| < \delta = \frac{1}{4}$$

$$\Rightarrow 0 < 2|x-3| < \frac{1}{2}$$

$$\Rightarrow 0 < |2x-6| < \frac{1}{2}$$

$$\Rightarrow 0 < |(2x+3)-9| < \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} < (2x+3)-9 < \frac{1}{2}$$

$$\Rightarrow (2x+3) < 9 + \frac{1}{10} < 10.$$

Therefore, your proof that

$$\lim_{x \to 2} 2x+3 = 10$$

cannot be correct.

The δ - ε Story

Bob: Okay, okay, I got it:

$$\lim_{x \to 2} 2x+3 = 9 + \frac{1}{100}$$

I even checked with a calculator.

Alice: Sorry, Bob, you are still wrong.

For $\varepsilon = \frac{1}{1000}$ we may take $\delta = \frac{1}{2000}$.

We get:

$$0 < |x-3| < \delta$$

$$\Rightarrow 0 < 2|x-3| < 2\delta$$

$$\Rightarrow 0 < |2x-6| < 2\delta$$

$$\Rightarrow 0 < |(2x+3)-9| < 2\delta$$

$$\Rightarrow 0 < |(2x+3)-9| < \frac{2}{2000} = \frac{1}{1000}$$

$$\Rightarrow -\frac{1}{1000} < |(2x+3)-9| < \frac{1}{1000}$$

$$\Rightarrow (2x+3)-9 < \frac{1}{1000}$$

$$\Rightarrow 2x+3 < 9 + \frac{1}{1000}$$

Therefore, "$\lim_{x \to 2} 2x+3 < 9 + \frac{1}{1000}$"

The δ-ε Story

Bob: Uhh-ohh. I am wrong again.

Alice: Yes — If you claim that

$$\lim_{x \to 2} 2x + 3 = c$$

for any value of $c \neq 9$

then I can produce an ε and δ

which will show that you are

wrong.

If you pick $c \neq 9$ then

I can find $\varepsilon < |c - 9|$

and use the value $\delta = \frac{\varepsilon}{2}$ to

show that your value was

incorrect.

* At that moment Bob understood *.