MAT A31 wk2a 

This week:

§1.1 & 1.2 : Limits.

"Limits are the backbone of calculus."

They are the first time we use infinity seriously.

Limits formalize the following ideas:

- As $x$ gets arbitrarily close to $c$, $f(x)$ gets arbitrarily close to $L$
- As $x$ gets arbitrarily large, $f(x)$ gets arbitrarily close to $L$.

$d(x,y) = |x-y|$

Defn: We write

$$\lim_{x \to c^-} f(x) = L$$

for the statement:

For all $\varepsilon > 0$:
There is $\delta > 0$:

$x \in (c-\delta, c) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

The idea being:
The values on the left of $x=c$ all get mapped close to $L$.

Defn: We write

$$\lim_{x \to c^+} f(x) = L$$

for the statement:

For all $\varepsilon > 0$
There is $\delta > 0$

$x \in (c, c+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

**Def$^n$:** We write

$$\lim_{x \to c} f(x) = L \text{ if:}$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

Equivalently,

$\lim_{x \to c} f(x) = L$ if

- For all $\varepsilon > 0$

 There is $\delta > 0$ s.t.:

$$x \in (c-\delta, c) \cup (c, c+\delta)$$

$$\implies f(x) \in (L-\varepsilon, L+\varepsilon)$$

Put in to words,

"If $\lim_{x \to c} f(x) = L$ then for any amount of error ($\varepsilon > 0$) there is a difference ($\delta > 0$) so that if $d(x,c) < \delta$ then $d(f(x), L) < \varepsilon$."

## Limits to Infinity.

"What does

$$\lim_{x \to \infty} f(x) = L$$

mean?"

For all $\varepsilon > 0$ there is $N = N(\varepsilon) > 0$ such that:

$$x > N \implies f(x) \in (L-\varepsilon, L+\varepsilon)$$

**Ex:** $\lim_{x \to \infty} \frac{1}{x} = 0$

**Pf:** Fix $\varepsilon > 0$.

If $N > \frac{1}{\varepsilon}$ then

$$\frac{1}{N} < \varepsilon.$$

Therefore,

if $x > N > \frac{1}{\varepsilon}$ we get

$$f(x) = \frac{1}{x} \in (0-\varepsilon, 0+\varepsilon)$$

Thus,

$$\lim_{x \to \infty} \frac{1}{x} = 0. \quad \text{QED.}$$

## operties of Limits

**rm** (Limits are unique)

f $\lim_{x \to a} f(x) = L$ and

$\lim_{x \to a} f(x) = M$

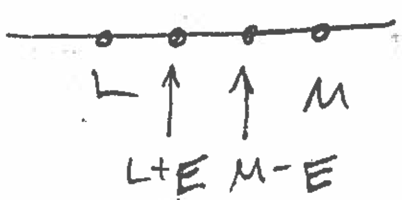
hen $L = M$.

:

ppose, for the sake of contradiction, that $L \neq M$.

e may assume $L < M$.

ick $E$ so that:




e get $L + E < M - E$,

**Pf** (con't):

From $\lim_{x \to c} f(x) = L$ we get:

For all $\varepsilon > 0$

There is $\delta_1 > 0$

$x \in (c - \delta_1, c) \cup (c, c + \delta_1)$

$\Longrightarrow f(x) \in (L - \varepsilon, L + \varepsilon)$

From $\lim_{x \to c} f(x) = M$ we get:

For all $\varepsilon > 0$

There is $\delta_2 > 0$

$x \in (c - \delta_2, c) \cup (c, c + \delta_2)$

$\Longrightarrow f(x) \in (M - \varepsilon, M + \varepsilon)$

By taking

$\varepsilon = E$ and

$\delta < \delta_1, \delta_2$

We get

$f(x) \in (M - E, M + E)$ and

$f(x) \in (L - E, L + E)$

a contradiction.

QED

## Properties of Limits (con't)

**Thm:**

$$\lim_{x\to c} f(x)=L \iff \begin{Bmatrix} \lim_{x\to c^+} f(x)=L \\ \lim_{x\to c^-} f(x)=L \end{Bmatrix}$$

**Pf:**

($\Longrightarrow$)

Suppose $\lim_{x\to c} f(x)=L$.

We have:

For all $\varepsilon>0$
There is $\delta>0$

$x\in(c-\delta,c)\cup(c,c+\delta)$
$\Longrightarrow f(x)\in(L-\varepsilon,L+\varepsilon)$

Thus,

For all $\varepsilon>0$
There is $\delta>0$ s.t.

$x\in(c-\delta,c) \Rightarrow f(x)\in(L-\varepsilon,L+\varepsilon)$

$$\left[\lim_{x\to c^-} f(x)=L\right]$$

etc.

($\Longleftarrow$)

Suppose

$\left(\lim_{x\to c^-} f(x)=L\right)$ $\left(\lim_{x\to c^+} f(x)=L\right.$

For all $\varepsilon>0$
There is $\delta_1>0$

For all $\varepsilon$
There is $\delta_2$

Pick $\delta<\min\{\delta_1,\delta_2\}$

We get:

For all $\varepsilon>0$
There is $\delta>0$

$x\in(c-\delta,c)\cup(c,c+\delta)$
$\Longrightarrow f(x)\in(L-\varepsilon,L+\varepsilon$

Consider the functions

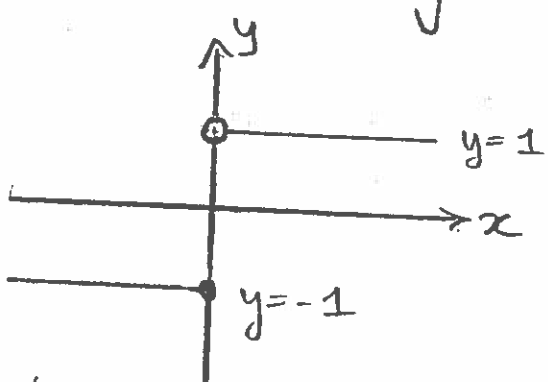
$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

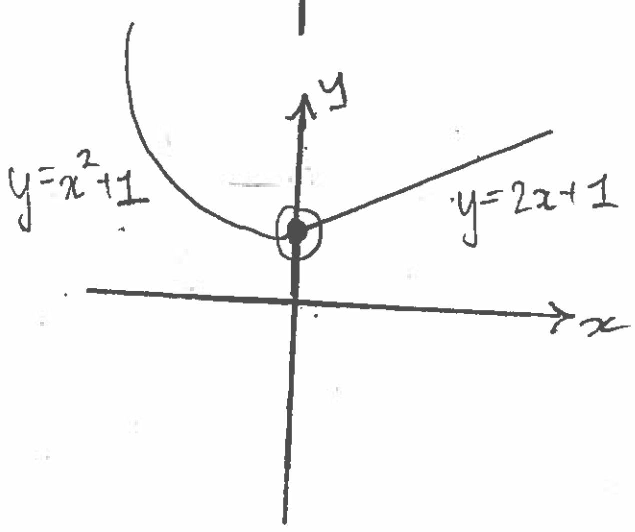
and $$g(x) = \begin{cases} 2x+1 & x \geq 0 \\ 1+x^2 & x < 0 \end{cases}$$

Discuss:

How do $f(x)$ ad $g(x)$ differ qualitatively near zero?

Ans: # Graph f and g.




In the case of $f(x)$:

The values on the left and right of $x=0$ differ.

$$\lim_{x\to 0^+} f(x) = 1, \quad \lim_{x \to 0^-} f(x) = -1.$$

In the case of $g$:

The values on the left and right of $x=0$ agree.

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = 1$$

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Consider the functions

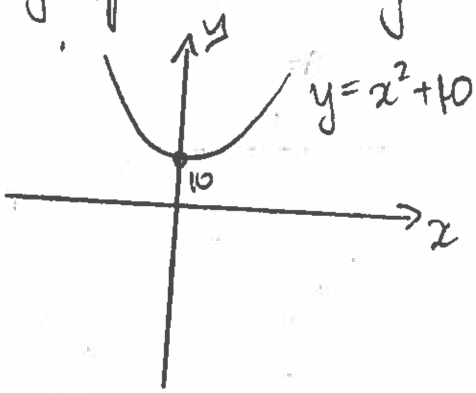
$$f(x) = x^2 + 10$$

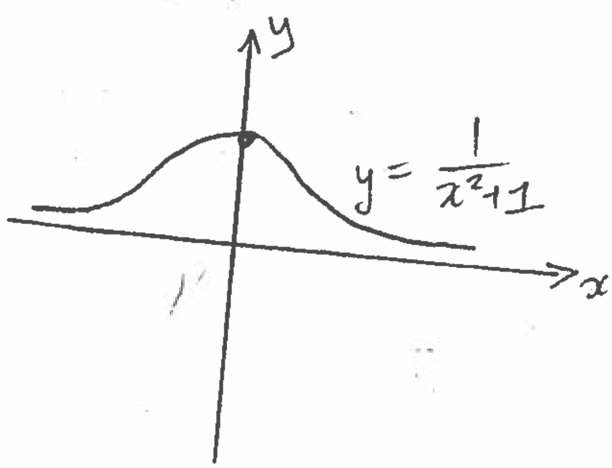
$$g(x) = \frac{1}{x^2+1}$$

**Discuss:**

How do $f(x)$ and $g(x)$ differ qualitatively when $x$ gets large?

Ans: # graph $f$ and $g$.




In the case of $f(x)$:

The function grows without bound

$$\lim_{x \to \infty} f(x) = \infty$$

In the case of $g(x)$:

The function is bound

It approaches a definite value.

(Horz. asymptote $y = ($

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Limits are used for two things:

- understanding local behaviour:

  "If $x$ is near $c$ then $f(x)$ is near $L$."

- understanding asymptotic behaviour;

  "If $x$ is large enough then $f(x)$ is near $L$."

In this lecture we will treat limits informally. Next lecture we will cover them formally.

Defn:

- $\lim_{x \to c^-} f(x)$ is the value $f(x)$ approaches when $x$ increases to $c$.
- $\lim_{x \to c^+} f(x)$ is the value $f(x)$ approaches when $x$ decreases to $c$.

Defn:

$f$ has a VERTICAL ASYMPTOTE if: one or more of the following hold;

$$\lim_{x \to c^-} f(x) = \pm\infty$$

$$\lim_{x \to c^+} f(x) = \pm\infty$$

$f$ has a HORIZONTAL ASYMPTOTE if:

$$\lim_{x \to \infty} = L \quad \text{or}$$

$$\lim_{x \to -\infty} = L \in \mathbb{R}$$

Ex: Suppose

$$f(x) = \begin{cases} e^{x+2} & x \geq 5 \\ 2x+3 & x < 5 \end{cases}$$

What are $\lim_{x \to 5^-} f(x)$

and $\lim_{x \to 5^+} f(x)$?

# To calculate $\lim_{x \to 5^-} f(x)$

use $f(x) = 2x+3$ $(x<5)$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} 2x+3 = 13$$

# To calculate $\lim_{x \to 5^+} f(x)$

use $f(x) = e^{x+2}$ $(x \geq 5)$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} e^{x+2} = e^7$$

NB:
the graph of $f(x)$ jumps from 13 to $e^7$ near $x=5$.

Ex: Suppose $x(n) = \frac{1}{10^n}$

What is

$\lim_{n \to \infty} x(n)$?

# Make a table of value:

| $n$ | $x(n) = 1/10^n$ |
|---|---|
| 1 | $1/10 = 0.1$ |
| 2 | $1/100 = 0.01$ |
| 3 | $1/1000 = 0.001$ |
| 4 | $1/10000 = 0.0001$ |

We guess, $\lim_{n \to \infty} \frac{1}{10^n} = 0$.

Ex: Suppose $x(1) = 1$,

$$x(n+1) = \frac{1}{x(n)+1}$$

What is

$\lim_{n \to \infty} x(n)$?

| $n$ | $x(n)$ |
|---|---|
| 1 | 1 |
| 2 | 2/1 |
| 3 | 3/2 |
| 4 | 5/3 = 1.66 |
| 5 | 13/8 = 1.625 |
| 100 | $\sim 1.618033\ldots$ |

$x = \frac{1}{x+}$