ε-δ Proofs (Intuition)

Recall, $\lim_{x \to c} f(x) = L \iff \forall_{\varepsilon > 0} \exists \delta > 0$:

$$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Error - Difference (Interpretation)

Given any amount of error $\varepsilon > 0$

There is an amount of difference $\delta > 0$

Such that:

"If the difference $|x-c|$ is at most δ

then the error $|f(x)-L|$ is at most ε."

Rise - Run (Interpretation)

Given any amount of rise $\varepsilon > 0$

There is an amount of run $\delta > 0$

such that:

"If the run $|x-c|$ is at most δ

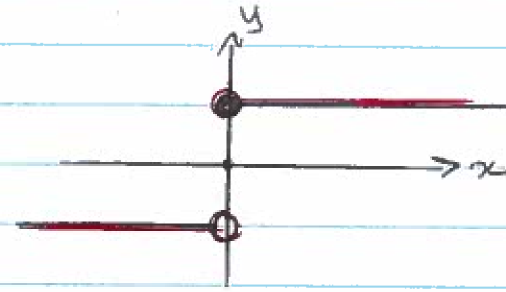
then the rise $|f(x)-L|$ is at most ε."

Ex: (An ε–δ proof that fails)

Suppose $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Claim: $\lim_{x \to 0} f(x) \neq 0$

Draw the graph of $f(x)$.



Suppose, for the sake of contradiction, that $\lim_{x \to 0} f(x) = 0$. We get:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : 0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \varepsilon.$$

The term "$\forall \varepsilon > 0$" dooms this example.

For any $\varepsilon < 1$ we encounter the following.

If $\varepsilon < 1$ then there is no $\delta > 0$ such that

$$0 < |x| < \delta \rightarrow |f(x)| < \varepsilon.$$

If $\delta > 0$ we get $f\left(\frac{\delta}{2}\right) = 1$ and $f\left(\frac{-\delta}{2}\right) = -1$.

Thus $\left|f\left(\frac{\delta}{2}\right) - 0\right| = 1 > \varepsilon$. Oh no!

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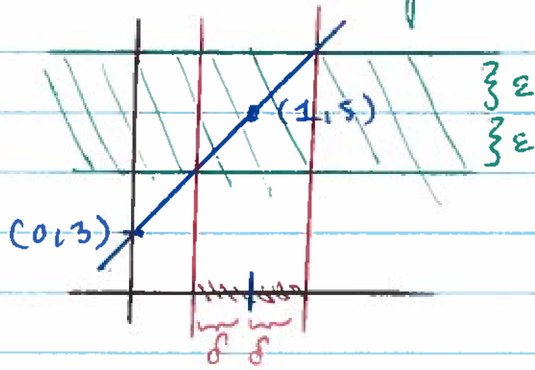
Worked Examples

Ex: $\lim_{x \to 1^+} 2x+3 = 5$

Want to show: $\forall_{\varepsilon>0} \exists_{\delta>0}\ x \in (1, 1+\delta) \Rightarrow |f(x)-5| < \varepsilon$

Draw a picture.



To ensure "rise $< \varepsilon$"
we take "run $< \frac{\varepsilon}{2}$"

$$|(2x+3)-5| = |2x-2|$$
$$= 2|x-1|$$
$$2|x-1| < \varepsilon \Leftrightarrow |x-1| < \frac{\varepsilon}{2}$$

Pf: Let $\varepsilon > 0$.
Pick $\delta < \frac{\varepsilon}{2}$.

$$\begin{aligned}
x \in (1, 1+\delta) &\Rightarrow 1 < x < 1+\delta \\
&\Rightarrow 2 < 2x < 2+2\delta \\
&\Rightarrow 2+3 < 2x+3 < 2+3+2\delta \\
&\Rightarrow 5 < 2x+3 < 5+2\delta \\
&\Rightarrow 0 < 2x+3-5 < 2\delta \\
&\Rightarrow |(2x+3)-5| < 2\delta \\
&\Rightarrow |(2x+3)-5| < 2\left(\tfrac{\varepsilon}{2}\right) = \varepsilon.
\end{aligned}$$

QED.

Ex: $\lim_{x \to 1^-} 6x - 1 = 5.$

Pf: Given $\varepsilon > 0$
Pick $\delta < \frac{\varepsilon}{6}$.

$$x \in (1-\delta, 1) \Rightarrow 1-\delta < x < 1$$
$$\Rightarrow 6-6\delta < 6x < 6$$
$$\Rightarrow 5-6\delta < 6x-1 < 5$$
$$\Rightarrow -6\delta < (6x-1)-5 < 0.$$
$$\Rightarrow |(6x-1)-5| < 6\delta$$
$$\Rightarrow |(6x-1)-5| < 6\left(\frac{\varepsilon}{6}\right) = \varepsilon$$

QED.

For $y = 2x+3$ we choose $\delta < \frac{\varepsilon}{2}$.

For $y = 6x-1$ we choose $\delta < \frac{\varepsilon}{6}$

Discuss: To prove $\lim_{x \to 1} f(x) = 5$

where $f(x) = \begin{cases} 6x-1 & x<1 \\ 2x+3 & x \geq 1 \end{cases}$

What δ should we pick?

Ans: Choose $\delta < \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{6}\right\}$.

This choice works for both $y = 2x+3$ AND $y = 6x-1$.

Constraining δ for convenience.

Ex: $\lim_{x \to 2} x^3 = 8$.

\# Need to show:
$\forall_{\varepsilon > 0} \exists_{\delta > 0}$:
$0 < |x-2| < \delta \Rightarrow |(x^3) - 8| < \varepsilon$.

Suppose $\delta < \frac{1}{10}$.

We get $2 - \frac{1}{10} < x < 2 + \frac{1}{10}$

$(1.9 < x < 2.1)$

$$|x^3 - 8| = |\underbrace{(x-2)}_{\text{controlled by } \delta}\underbrace{(x^2+2x+4)}_{\text{uncontrolled}}|$$

To show $|x^3 - 8| < \varepsilon$ we need to control both parts.

It follows:

$$(1.9)^2 + 2(1.9) + 4 < x < (2.1)^2 + 2(2.1) + 4$$
$$(11.41 < x < 12.61)$$

Thus, we take $\delta < \min\left\{\frac{1}{10}, \frac{\varepsilon}{12.61}\right\}$

Pf: Let $\varepsilon > 0$.
Take $\delta < \min\left\{\frac{1}{10}, \frac{\varepsilon}{12.61}\right\}$

$$0 < |x-2| < \delta \Rightarrow 0 < |x-2|\,|x^2+2x+4| < \delta|\cdots|$$
$$\Rightarrow 0 < |x^3 - 8| < \delta\,|x^2+2x+4|$$
$$\Rightarrow 0 < |x^3-8| < \left(\frac{\varepsilon}{12.61}\right)(12.61) = \varepsilon.$$

$\delta < \frac{\varepsilon}{12.61}$. $\delta < \frac{1}{10}$

QED.

Infinity and Limits

Discuss: What is the quantified version of $\lim_{x \to c} f(x) = \infty$

Ans: For all $N > 0$:
There is $\delta > 0$:
(p94)
$0 < |x - c| < \delta \Rightarrow N < f(x)$

NB: No instance of ∞ appears

Ex: $\lim_{x \to 0} \frac{1}{x^2} = \infty$

given $N = 100$

$\delta < \frac{1}{10}$ works $\frac{1}{\left(\frac{1}{10}\right)^2} = 100$

given $N = 10000$

$\delta < \frac{1}{100}$ works $\frac{1}{\left(\frac{1}{100}\right)^2} = 10000$

given $N = 1000\,000$

$\delta < \frac{1}{1000}$ works. $\frac{1}{\left(\frac{1}{1000}\right)^2} = 1\,000\,000$

Pf: Fix $N > 0$.

Take $\delta < \frac{1}{\sqrt{N}}$

$0 < |x| < \delta < \frac{1}{\sqrt{N}} \Rightarrow \frac{1}{x^2} > N$ QED.

Discuss: What is the quantified version of

$$\lim_{x\to\infty} f(x) = \infty$$

Ans: (p94) For all $N > 0$:
There is $M > 0$:
$M < x \implies N < f(x)$

NB: No instance of the symbol ∞ appears.

Ex: $\lim_{x\to\infty} \ln(x) = \infty$

given $N = 100$
$M > e^{100}$ works. $\ln(e^{100}) = 100$

given $N = 1000$
$M > e^{1000}$ works. $\ln(e^{1000}) = 1000$

Pf: Fix $N > 0$
Pick $M > e^N$

$$x > M \implies x > e^N$$
$$\implies \ln(x) > N$$

QED.

Discuss: What is the quantified version of

$$\lim_{x \to -\infty} f(x) = L.$$

Ans: For all $\varepsilon > 0$:
There is $N < 0$:

Play around with $-\infty$

$$x < N \implies |f(x) - L| < \varepsilon.$$

Ex: $\lim_{x \to -\infty} 10^x = 0$.

Given $\varepsilon = 1$

$10^0 = 1$

$N < 0$ works.

Given $\varepsilon = \frac{1}{10}$

$10^{-1} = \frac{1}{10}$

$N < -1$ works.

Given $\varepsilon = \frac{1}{100}$

$10^{-2} = \frac{1}{100}$

$N < -2$ works.

Pf: Fix $\varepsilon > 0$

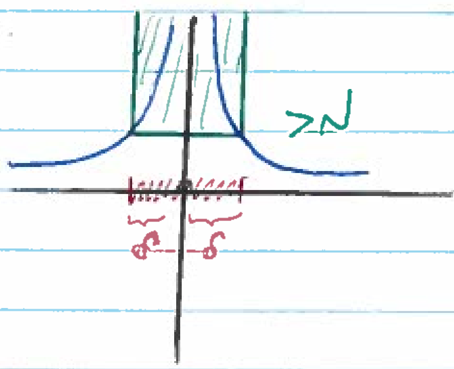
Pick $N < \log(\varepsilon)$.

$x < N < \log(\varepsilon)$

$\Rightarrow 10^x < 10^{\log(\varepsilon)}$

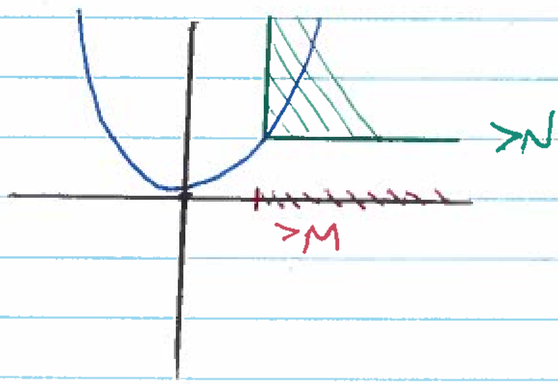
$\Rightarrow 10^x < \varepsilon$. QED.

Model Limits



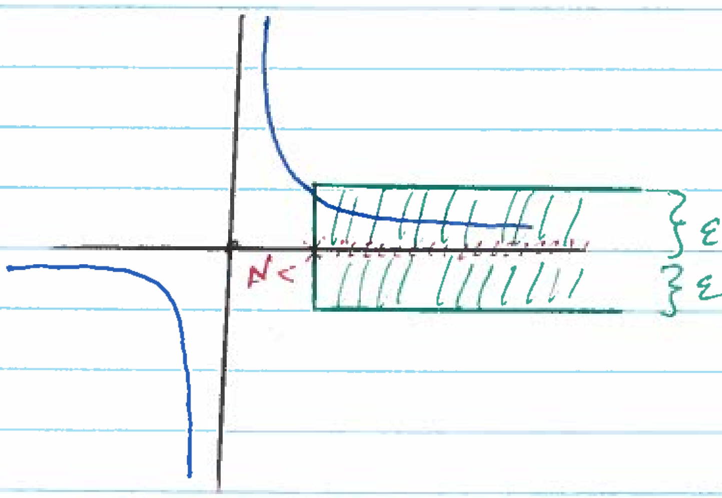
$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

∞ is an amount of bigness



$$\lim_{x\to\infty} x^2 = \infty$$

∞ means "arbitrarily large"



$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

If you can: ① Remember/Invent the correct quantifiers
② Prove these limits

then all is well.

The Art of Estimates (1)

Ex: $\lim_{x \to \infty} \left(\frac{1}{x}\right)\sin(x) = 0$ Is sin(x) important?

[redacted] We know $-1 \leq \sin \leq 1$.
Thus, we may "ignore" it.

Given $\varepsilon = \frac{1}{10} = 0.1$

$N > 10$ works

Given $\varepsilon = \frac{1}{100} = 0.01$

$N > 100$ works

Given $\varepsilon = \frac{1}{1000} = 0.001$.

$N > 1000$ works.

Pf: Given $\varepsilon > 0$.

Choose $N > \frac{1}{\varepsilon}$.

$x > N \implies \frac{1}{x} < \frac{1}{N} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$.

$\implies \frac{1}{x} < \varepsilon$

$\implies \left|\frac{1}{x}\sin(x)\right| < \varepsilon$. QED.