V3

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Bounds

Defn: Let $S \subseteq \mathbb{R}$ be a non-empty set of reals

N is an UPPER BOUND for S if:

$$x \in S \implies x \leq N$$

l is a LOWER BOUND for S if:

$$x \in S \implies l \leq x$$

A set can have upper or lower bounds.

Discuss: Complete the following:

		UPPER Yes	UPPER No
LOWER	Yes	$\{1,2,3\}$	$[\pi, \infty)$
LOWER	No	$(-\infty, e)$	$(-\infty, \infty)$

Fact:
- If N is an upper bound for S and $N \leq N'$ then N' is an upper bound for S.
- If l is a lower bound for S and $l' \leq l$ then.

What are the "best" bounds?

Axiom: If $S \subseteq \mathbb{R}$ is not empty and S has an upper bound then there is a least upper bound lub(S) such that:

If S is bounded above by N then $\text{lub}(S) \leq N$

Axioms are propositions that we assume without proof. "The rules of the game."

Def^n: If S is bounded above then lub(S) is an upper bound such that:

[If N is an upper bound for S $\text{lub}(S) \leq N$

Def^n: If S is bounded below then the greatest lower bound glb(S) is such that: [If l is a lower bound for S then $l \leq \text{glb}(S)$

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Ex: Compute $lub(S)$ and $glb(S)$ for $S = \{1, 2, 3\}$

compute $lub(S)$.

Clearly $x \in S \Rightarrow x \leq 3$
Thus, 3 is an upper bound for S.

By defn of lub, $3 \geq lub(S)$

Thus $3 = lub(S)$ OR $3 > lub(S)$.

If $3 > lub(S)$ then $lub(S)$ is not an upper bound for S since $3 \in S$.

Thus, $lub(S) = 3$.

compute $glb(S)$

Clearly, $x \in S \Rightarrow 1 \leq x$.
Thus, 1 is a lower bound for S.
Therefore, $1 \leq glb(S)$.

If $1 < glb(S)$ then $glb(S)$ is not a bound for S since $1 \in S$.

Thus $1 = glb(S)$.

Ex: compute glb(S) and lub(S) when $S = [2,3) = \{x : 2 \leq x < 3\}$

closed open

compute glb(S)

$x \in S \implies 2 \leq x$

Thus, 2 is a lower bound for S.

By defn: $2 \leq glb(S)$ → $2 < glb(S)$
↘ $2 = glb(S)$.

If $2 < glb(S)$ then glb(S) is not a lower bound for S since $2 \in S$. (!) We use $2 \in S$.

Thus, $glb(S) = 2$.

compute lub(S)

$x \in S \implies x < 3 \implies$ 3 is an upper bound
$\implies lub(S) \leq 3$.
$\implies lub(S) = 3$ or $lub(S) < 3$.

If $lub(S) < 3$ then $lub(S) < 3 - \frac{1}{10^n} < 3$ (for some n.)
(!) We use $3 - \frac{1}{10^n}$ to name the unbounded element.
Thus lub(S) is not a bound for S since $3 - \frac{1}{10^n} \in S$ for all S.

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Ex: Compute glb(S) for $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

$= \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Observe $0 < \frac{1}{n}$ for all n.
Thus $0 \leq glb(S)$
Either, $0 = glb(S)$ or $0 < glb(S)$.

If $0 < glb(S)$ then

$0 < \frac{1}{n} < glb(S)$ for some n.

Thus, glb(S) is not a lower bound for S. (Contradiction)

Therefore, $glb(S) = 0$.

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Continuity

Def^n: A function $f: \mathbb{R} \to \mathbb{R}$ is CONTINUOUS

if: $\lim_{x \to c} f(x) = f(c)$ for all $c \in \mathbb{R}$.

Ex: The constant functions are continuous.

In a certain sense they are the MOST continuous!

Let $f(x) = k$ for all x.

clearly, $\lim_{x \to c} k = k = f(c)$.

Ex: Linear functions are continuous.

Let $f(x) = mx + b$, $(m, b \in \mathbb{R})$

$\left[\lim_{x \to c} f(x) = f(c) \right.$

$\Big[$ For all $\varepsilon > 0$:

There is $\delta > 0$:

$0 < |x - c| < \delta \implies |(mx+b) - (mc+b)| < \varepsilon$.

For $\varepsilon > 0$, take $\delta < \frac{\varepsilon}{|m|}$.

Thus, linear functions are continuous.

Ex: The function $f(x)=x^2$ is continuous.

Pf: Fix $c \in \mathbb{R}$. We show $\lim_{x\to c} f(x)=c^2$.

Want to show: $\lim_{x\to c} x^2 = c^2 \iff \left[\forall \varepsilon>0\ \exists \delta>0 : 0<|x-c|<\delta \implies |x^2-c^2|<\varepsilon \right.$

Take $\delta < \min\left\{\frac{1}{10}, \frac{\varepsilon}{\|2c+\frac{1}{10}\|}\right\}$

$0<|x-c|<\delta \implies 0<|x-c|\cdot|x+c|<\delta|x+c|$

$\implies 0<|x^2-c^2|<\frac{\varepsilon}{\|2c+\frac{1}{10}\|}\cdot\|2c+\frac{1}{10}\|$

$\implies 0<|x^2-c^2|<\varepsilon.$

QED.

Therefore, $\lim_{x\to c} x^2=c^2$ for all c.

It follows $f(x)=x^2$ is continuous.

continuity proofs require that

$\lim_{x\to c} f(x)=f(c)$ for all $c \in \mathbb{R}$

Examples without proofs:

- $\sin(x)$, $\cos(x)$
- e^x
- $\ln(x)$, $\log(x)$ $[x > 0]$
- all polynomials.

Intuition

A function is continuous if its graph is a smooth curve. with no jumps.

If $f(x)$ is continuous and

: $f(a) < 0$ then there is an interval $(a-\delta, a+\delta)$ such that $x \in (a-\delta, a+\delta) \Rightarrow f(x) < 0$.

Pf: Take $\varepsilon < \dfrac{|0 - f(x)|}{2}$. There is $\delta > 0$ by continuity.

QED.

Thm (The Intermediate Value Thm)

Let $a<b$ and $f(x)$ be continuous.

If $f(a) < 0 < f(b)$ then there is

$c \in (a,b)$ such that $f(c) = 0$.

This theorem is magic!

Pf: By the Lemma, there is an interval $(a-\delta, a+\delta)$ such that $x \in (a-\delta, a+\delta) \Rightarrow f(x) < 0$.

- Let $S = \{x : f(x) < 0\}$. NB: $S \neq \emptyset$.

  We know that S is bounded above by b.

  Thus, S has a least upper bound.

- Let $x_0 = \text{lub}(S)$. NB: $x_0 \in (a,b)$.

  Consider $f(x_0)$. Either $f(x_0) < 0$, $f(x_0) = 0$ OR $f(x_0) > 0$.

Pf (con't)

- If $f(x_0) < 0$ then we may apply the Lemma to get $(x_0 - \delta, x_0 + \delta)$ such that $x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) < 0$.

$$\text{Thus, } \operatorname{lub}(S) \geq x_0 + \delta$$
$$x_0 \geq x_0 + \delta$$
$$0 \geq \delta \quad \text{(contradiction!)}$$

- If $f(x_0) > 0$ then there is an interval $(x_0 - \delta, x_0 + \delta)$ such that:

$$x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) > 0.$$

$$\text{Thus, } \operatorname{lub}(S) \leq x_0 - \delta$$
$$x_0 \leq x_0 - \delta$$
$$0 \leq -\delta$$
$$\delta \leq 0 \quad \text{(contradiction!)}$$

Therefore, $f(x_0) = 0$ as required. QED.