Logarithms/Exponents and Derivatives

Ex: Find $\frac{dy}{dx}$ if $y = 3^x$

Express 3^x in terms of e^x

$$y = [e^{\ln(3)}]^x = e^{[x \cdot \ln(3)]}$$

Apply the chain rule

$$\frac{dy}{dx} = \frac{d\, e^{[x \cdot \ln(3)]}}{dx}$$

$$= \frac{d\, e^{[x \cdot \ln(3)]}}{d[x \cdot \ln(3)]} \cdot \frac{d[x \cdot \ln(3)]}{dx}$$

$$= e^{[x \cdot \ln(3)]} \cdot \ln(3)$$

$$= \ln(3) \cdot 3^x$$

Thm: $f(x) = a^x \quad (a > 0, a \neq 1)$

$$\Longrightarrow f'(x) = a^x \cdot \ln(a)$$

Tool: $a^x = [e^{\ln(a)}]^x$

Defn: $a^t = b \iff t = \log_a(b)$

The "log base a of b" solves $a^t = b$.

NB: $\log_a(x)$ is the inverse of a^x.

Ex: Find $\frac{dy}{dx}$ if $y = \log_{10}(x)$

\# Use the inverse function

$$10^{[\log_{10}(x)]} = x$$

$$\Rightarrow \frac{d\,10^{[\log_{10}(x)]}}{d[\log_{10}(x)]} \cdot \frac{d\log_{10}(x)}{dx} = 1$$

$$\Rightarrow \left(10^{\log_{10}(x)} \cdot \ln(10)\right) \cdot \frac{d\log_{10}(x)}{dx} = 1$$

$$\Rightarrow x \cdot \ln(10) \cdot \frac{d\log_{10}(x)}{dx} = 1$$

$$\Rightarrow \frac{d\log_{10}(x)}{dx} = \frac{1}{x \cdot \ln(10)}$$

Thm: $f(x) = \log_a(x) \quad (a > 0, a \neq 1)$

$$\Rightarrow f'(x) = \frac{1}{\ln(a) \cdot x}$$

Logarithmic Differentiation

Ex: Find $\frac{dy}{dx}$ if $y = x^x$

NB: We are stuck! It is neither poly nor exp!

Bring the exponent down using $\ln(x)$.

$$\ln(y) = \ln(x^x) = x\ln(x)$$

$$\frac{d}{dx}[\ln(u)] = \frac{d}{dx}[x\ln(x)]$$

$$\frac{d\ln(y)}{dx} = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1.$$

$$\frac{d\ln(y)}{dy} \cdot \frac{dy}{dx} = \ln(x) + 1$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(x) + 1$$

$$\frac{dy}{dx} = x^x(\ln(x) + 1).$$

Ex: Apply logarithmic differentiation to:

$$y = \frac{\sqrt{x}\,(x^2-1)^5}{(x+2)\cdot(x-4)^3}$$

$$\ln(y) = \ln\left(\frac{\sqrt{x}\,(x^2-1)^5}{(x+2)(x-4)^3}\right)$$

$$= \ln(\sqrt{x}) + \ln([x^2-1]^5) - \ln(x+2) - \ln([x-4]^3)$$

$$= \frac{1}{2}\ln(x) + 5\ln(x^2-1) - \ln(x+2) - 3\ln(x-4)$$

differentiate both sides.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x} + \frac{5\cdot 2x}{x^2-1} - \frac{1}{x+2} - \frac{3}{x-4}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} + \frac{10x}{x^2-1} - \frac{1}{x+2} - \frac{3}{x-4}\right)$$

$$= \left[\frac{\sqrt{x}\,(x^2-1)^5}{(x+2)(x-4)^3}\right]\left(\frac{1}{2x} + \frac{10x}{x^2-1} - \frac{1}{x+2} - \frac{3}{x-4}\right)$$

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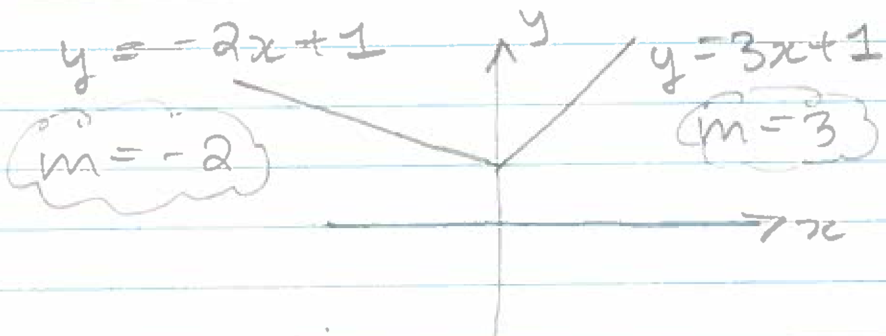
Piecewise Differentiation

Ex: Find $\frac{dy}{dx}$ if $y = \begin{cases} 3x+1 & x \geq 0 \\ -2x+1 & x < 0 \end{cases}$.



$$\frac{dy}{dx} = \begin{cases} 3 & x > 0 \\ 2 & x < 0 \\ \text{undef} & x = 0 \end{cases}$$

Ex: Find $\frac{dy}{dx}$ if $y = \ln(|x|) = \begin{cases} \ln(x) & x > 0 \\ \ln(-x) & x < 0 \end{cases}$.

If $x > 0$ then $\frac{dy}{dx} = \frac{d\ln(x)}{dx} = \frac{1}{x}$.

If $x < 0$ then $\frac{dy}{dx} = \frac{d\ln(-x)}{dx} = \frac{1}{-x}(-1)$

Thus, $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$.

This week: §2.5 Derivatives of Exp/Log

§2.6 Derivatives of Trig/Hyp. Trig

The Functional Equation $f(x+y)=f(x)f(y)$.

This has the "boring" solution $f(t) \equiv 0$.

Suppose $f(t) \neq 0$.

We get: $f(t) = f(t+0) = f(t)f(0)$

$$\Longrightarrow f(0) = \frac{f(t)}{f(t)} = 1.$$

We also get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= f(x) \cdot f'(0).$$

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Fact: $f(x)=e^x$ is the unique function such that:
- $f'(x)=f(x)$
- $f'(0)=1$

(Unfortunately, we cannot prove this.)

Fact: $f'(x)=k\,f(x) \iff f(x)=C\cdot e^{k\cdot x}$

Exponential functions are the only functions proportional to their own derivative.

Detour towards Proof

If $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\dots$

then $f'(x)=a_1+2a_2x+3a_3x^2+\dots$

Thus — $f'(x)=f(x)$ $\qquad$ $f'(0)=1$

$$\iff \begin{aligned} a_0&=a_1\\ a_1&=2a_2\\ a_2&=3a_3\\ &\vdots \end{aligned} \qquad \Rightarrow a_1=1$$

Therefore, $2\cdot(a_2)=1 \Rightarrow a_2=\frac{1}{2}$ $\qquad$ $3\cdot a_3=a_2 \Rightarrow a_3=\frac{1}{2\cdot 3}$ $\qquad$ $a_n=\frac{1}{n!}$

We get $f(x)=1+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

Exercise: Show $f(x+y)=f(x)f(y)$ using algebra.

A31 Wk 9a

Derivatives of Inverse Functions

Defn: $g(x)$ is the INVERSE of $f(x)$ if:

$$g(f(x)) = f(g(x)) = x$$

Thm: If $f(x)$: Domain$(f) \to$ Range(f) is injective then it has an inverse $g(x)$: Range$(f) \to$ Domain(f).

Defn: The inverse of $f(x) = e^x$ is $g(x) = \ln(x)$.
"lawn of x"

Thm: If $f(x)$ and $g(x)$ are inverses:

$$g'(x) = \frac{1}{f'(g(x))}.$$

Pf: $f(g(x)) = x$

$$\Rightarrow \frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x]$$

$$\Rightarrow f'(g(x)) \cdot g'(x) = 1$$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))}.$$

Ex: Calculate $\frac{dy}{dx}$ if $y = \ln(x)$. $e^{\ln(x)} = x$

Ex: Find a function so that $f'(x) = 10f(x)$ and $f(0) = 5$.

Apply the classification proportional $\Rightarrow f(x) = C \cdot e^{kx}$

$$f(x) = c \cdot e^{kx} \Rightarrow f'(x) = k \cdot c \cdot e^{kx} = k f(x)$$
$$\Rightarrow k = 10.$$

$$f(0) = c \cdot e^{10 \cdot 0} = c \cdot e^{0} = C = 5$$

Thus, $f(x) = 5 \cdot e^{10x}$.

Ex: Find $\frac{dy}{dx}$ if $f(x) = \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{d \ln(x^2+1)}{dx} = \frac{d \ln(x^2+1)}{d(x^2+1)} \cdot \frac{d(x^2+1)}{dx}$$

$$= \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

Ex: Find $\frac{dy}{dx}$ if $y = x^x$.

$$\ln(y) = x \ln(x)$$
$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1$$
$$\Rightarrow \frac{dy}{dx} = x^x [\ln(x) + 1]$$