MAT 133 – Week 1c

We introduce the exponential functions and the rules for manipulating exponents.

Defⁿ: $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n\text{ - terms}}$

is the number a to the power n.

Ex: $3^3 = 3 \cdot 3 \cdot 3 = 27$

Facts: ① $a^n \cdot a^m = a^{n+m}$

② $(a^n)^m = a^{(nm)}$

③ $a^n / a^m = a^{n-m}$

Defⁿ: If $a \geq 0$ and n is a whole number then

$\sqrt[n]{a}$ is a non-negative b

such that:

$b^n = a$

Ex: $\sqrt[3]{8} = 2$ since $2^3 = 2 \cdot 2 \cdot 2 = 8$.

Fact: $\sqrt[n]{x} = x^{\frac{1}{n}}$ since $(x^{\frac{1}{n}})^n = x^{\frac{1}{n} \cdot n} = x^1 = x$.

(!) $\sqrt{x}$ is not defined when $x < 0$

(!) There are multiple solutions to $b^2 = 4$
For example, $b = +2$ and $b = -2$

We define $\sqrt{4} = +2$.

We introduce the logarithm function.

Def^n: If a and b are positive then

$$\log_a(b) = c \text{ such that } a^c = b.$$

In plain terms, the logarithm answers the question:

"To what power do I need to raise a in order to get b?"

Ex: $\log_{10}(1000) = 3$ $\quad \log_3(81) = 4$.

Facts: $\log_a(1) = 0$

$\log_a(a) = 1$

$\boxed{\log_a(a^x) = x}$ $\quad \boxed{a^{\log_a(b)} = b}$

These last two facts say: $y = \log_a(x)$ is the inverse of the function $y = a^x$.

Fact: $\log_a(xy) = \log_a(x) + \log_a(y)$

If $a^{\log_a(x)} = x$ and $a^{\log_a(y)} = y$ then

$$a^{\log_a(x) + \log_a(y)} = a^{\log_a(x)} \cdot a^{\log_a(y)}$$

$$= x \cdot y$$

Thus, $\log_a(xy) = \log_a(x) + \log_a(y)$

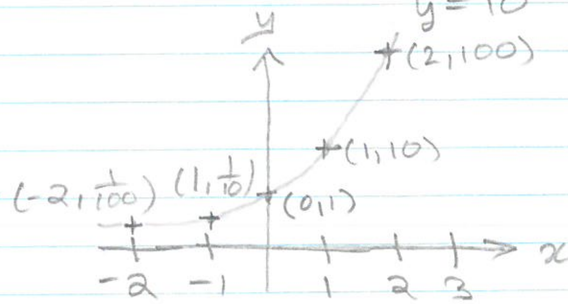
$$100 \cdot 1000 = 10^2 \cdot 10^3 = 10^5$$

We plot the graphs of the exponential and logarithm functions.

<u>Ex</u>: Plot $y = 10^x$.

Make a table of values.

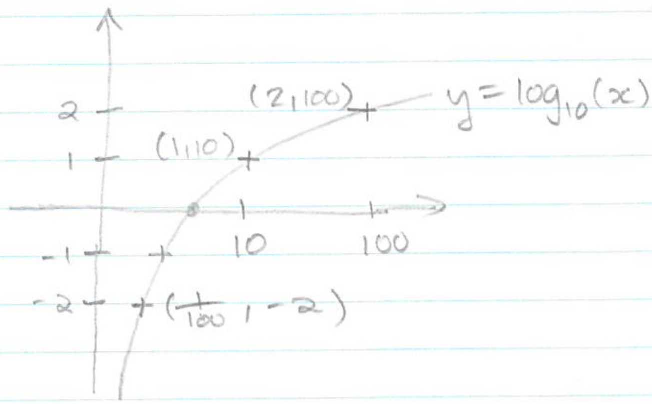
x	$y = 10^x$
-2	$\frac{1}{100}$
2	100



Plot $y = \log_{10}(x)$

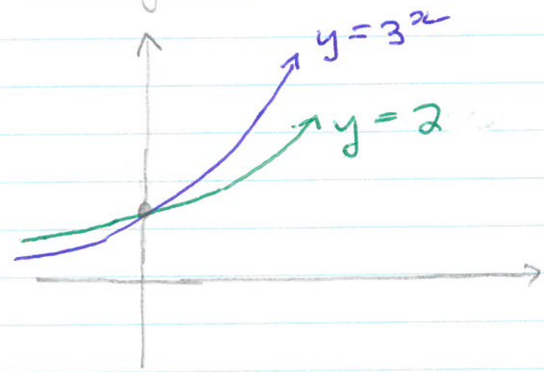
Make a table of values.

x	$y = \log_{10}(x)$
$\frac{1}{100}$	-2
$\frac{1}{10}$	-1
1	0
10	1
100	2



<u>Ex</u>: Plot the graphs $y = 2^x$ ad $y = 3^x$.

x	$y = 2^x$	$y = 3^x$
-2	$\frac{1}{4}$	$\frac{1}{9}$
-1	$\frac{1}{2}$	$\frac{1}{3}$
0	1	1
1	2	3
2	4	9



! For negative values $y = 3^x$ is smaller than $y = 2^x$
For positive values $y = 3^x$ is bigger than $y = 2^x$.

It is interesting to note that there is a function

$$y = (2.1)^x \text{ between } y = 2^x \text{ ad } y = 3^x.$$

We calculate using logs and exponentials.

Ex: Calculate

$$\log_2(4^5) = \log_2((2^2)^5) = \log_2(2^{10}) = 10$$

Ex: Calculate $\log_3(\sqrt[5]{3}) = \log_3(3^{\frac{1}{5}}) = \frac{1}{5}$

Ex: Simplify $\log_{10}\left(\frac{\sqrt{3}}{\sqrt[7]{12}}\right)$

$$= \log_{10}(\sqrt{3}) - \log_{10}(\sqrt[7]{12})$$

$$= \frac{1}{2}\log_{10}(3) - \frac{1}{7}\log_{10}(12).$$

Ex: Calculate $\frac{36}{\sqrt{6}} = \frac{6^2}{6^{\frac{1}{2}}} = 6^{\frac{3}{2}}$

Summary:

$a^n a^m = a^{n+m}$ $\qquad \log(xy) = \log(x) + \log(y)$

$(a^n)^m = a^{nm}$ $\qquad \log_a(a^x) = x$

$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\qquad a^{\log_a(x)} = x$