MAT 246 Wk 2a

Induction (Chpt. 2)

Thm: The Principle of Mathematical Induction states:

If S is a set of natural numbers such that:

A $1 \in S$

B $k \in S \Rightarrow k+1 \in S$

then $S = \mathbb{N}$.

Ex: Suppose S satisfies 1 and 2. Show that $4 \in S$.

By A we know $1 \in S$.
Thus, by B, $2 \in S$. [since $1 \in S \Rightarrow 2 \in S$]
Again, by B, $3 \in S$ [since $2 \in S \Rightarrow 3 \in S$]
Again, by B, $4 \in S$ [since $3 \in S \Rightarrow 4 \in S$.]

Therefore, $4 \in S$.

NB: Using this kind of argument we could show that any natural number is in S.

For example, we could show $42 \in S$ using A once and B fourty-one times.

Thm: For any natural n we have:

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Pf: Let S be the set of naturals for which the statement holds.

$$S = \{n \in \mathbb{N} : 1+2+\dots+n = \frac{n(n+1)}{2}\}$$

We note that $1 \in S$ since

$$LHS = 1 \text{ and } RHS = \frac{1(1+1)}{2} = 1$$

Suppose $k \in S$. We show $k+1 \in S$.

$$LHS = 1+2+3+\dots+k+(k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2}$$

$$RHS = \frac{(k+1)[(k+1)+1]}{2}$$

Thus $k+1 \in S$. By induction, $S = \mathbb{N}$. QED

Thm: For any integer $n \geq 0$ we have:

$$2^0 + 2^1 + 2^2 + \dots + 2^n = 2^{n+1} - 1$$

Pf: Let S be set of integers for which the statement holds:

$$S = \{ n \in \mathbb{Z} : 2^0 + 2^1 + \dots + 2^n = 2^{n+1} - 1 \}$$

We note that $0 \in S$ since:

$LHS = 2^0 = 1$ and $RHS = 2^{0+1} - 1 = 2 - 1 = 1$

Suppose $k \in S$. We show that $k+1 \in S$.

$$LHS = 2^0 + 2^1 + \dots + 2^k + 2^{k+1}$$

$$= (2^{k+1} - 1) + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$$= 2^{(k+1)+1} - 1$$

$$RHS = 2^{(k+1)+1} - 1$$

Thus, $k+1 \in S$. By induction, $S = \mathbb{N}$. QED

<u>Nota</u>: Recall, $k! = k\cdot(k-1)(k-2)\cdots 3\cdot 2\cdot 1$.

<u>Thm</u>: If $n \geq 4$ then $n! > 2^n$.

<u>Pf</u>. For $n=4$ we have:

$$4! = 4\cdot 3\cdot 2\cdot 1 = 24.$$

and $2^4 = 2\cdot 2\cdot 2\cdot 2 = 16$

Thus, $4! > 2^4$.

Suppose $k! > 2^k$ for $k \geq 4$.
We show $(k+1)! > 2^{k+1}$.

$$(k+1)! = k!\cdot(k+1)$$

$$> 2^k \cdot 4 \qquad \# \; k \geq 4 \Rightarrow k+1 > 4.$$

$$> 2^{k+2} = 2^{(k+1)+1}$$

Thus, $(k+1)! > 2^{(k+1)+1}$ and the claim follows by induction.

<u>Def^n</u>: The Generalized Principle of Induction

If S is a set of naturals such that:
1. $m \in S$
2. $k \in S \Rightarrow k+1 \in S$

then $S = \{m, m+1, m+2, \ldots\}$

<u>Defn</u>: The Fibonacci numbers are defined by:

$$F_1 = 1, \; F_2 = 1, \; F_n = F_{n-1} + F_{n-2}.$$

<u>Ex</u>: $F_3 = 1+1 = 2$, $F_4 = 2+1 = 3$, $F_5 = 3+2 = 5$.

<u>Thm</u>: For any n: $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

<u>Pf</u>: For $n=1$ we have:

$$RHS = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1.$$

For $n=2$ we have:

$$RHS = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+2\sqrt{5}+5}{4} - \frac{1-2\sqrt{5}+5}{4}\right]$$

$$= 1$$

<u>NB</u>: $x^2 = x+1 \Rightarrow x^2 - x - 1 = 0$

$$\Rightarrow x = \frac{1+\sqrt{5}}{2} \text{ OR } x = \frac{1-\sqrt{5}}{2}$$

We write $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$.

Pf (con't):

For $n \geq 3$ we have:

$$F_n = F_{n-1} + F_{n-2}$$

$$= \frac{1}{\sqrt{5}}\left[\varphi^{n-1} + \bar{\varphi}^{n-1}\right]$$

$$+ \frac{1}{\sqrt{5}}\left[\varphi^{n-2} + \bar{\varphi}^{n-2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\varphi^{n-1} + \varphi^{n-2} + \bar{\varphi}^{n-1} + \bar{\varphi}^{n-2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\varphi^{n-2}(\varphi+1) + \bar{\varphi}^{n-2}(\bar{\varphi}+1)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\varphi^{n-2}(\varphi^2) + \bar{\varphi}^{n-2}(\bar{\varphi}^2)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\varphi^n + \bar{\varphi}^n\right]$$

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well-ordering

Thm: The well-ordering Principle states:
If T is a non-empty set of natural numbers then T has a least element.

Ex: The least element of $\{2, 4, 6\}$ is 2.
The least element of $\{3, 5, 7, 9, 11, \ldots\}$ is 3.

We write WOP for "well ordering principle"
PMI for "principle of math. induction".

Thm: WOP $\Rightarrow$ PMI

Suppose the WOP holds, and PMI does NOT.
Let S be a set such that:
(A) $1 \in S$
(B) $k \in S \Rightarrow k+1 \in S$
and $S \neq \mathbb{N}$.

We let $T = \{n \in \mathbb{N} : n \notin S\}$ NB: $t \notin S$
T is not empty by our hypothesis on S.
Thus, by WOP, T has a least element $t \in T$.

We know $t \neq 1$ since $1 \in S$.
Thus $t \geq 2$.

Note that $t-1 \in S$, by the minimality of t.
Thus $t = (t-1)+1 \in S$.
Thus, $t \notin T$, contradicting our hypothesis on t.

Thm: PMI $\Rightarrow$ WOP

Pf: Let $S = \{ n \in \mathbb{N}$: If T is a non-empty set of naturals ad there is $t \in T$ such that $t \leq n$ then T has a least element$\}$

We note that:
If there is $t \leq 1$ in T then $1 \in T$ and 1 is the least element of T.

Suppose that if T is non-empty and there is $t \in T$ such that $t \leq k$ then T has a least element.

If $k+1 \in T$ then either:
$k+1 \leq t$ for all $t \in T$
$\Rightarrow k+1$ is the least element of T

OR

there is $t \in T$ such that $t < k+1$
$\Rightarrow t \leq k$ and hence T has a least element.

Thus, $S = \mathbb{N}$ ad we get:
If T is non-empty then it has a least element.

<u>Thm</u>: There is no natural number $0<n<1$.

<u>Pf</u>: Suppose $T=\{n\in\mathbb{N} : 0<n<1\}$ is non-empty.

By the well-ordering principle T has a least element n.

We obtain: $0<n$ and $n<1$

Thus,

$$0 < n\cdot n = n^2 \quad \text{and}$$

$$n\cdot n < 1\cdot n = n$$

We obtain, $0<n^2<n<1$.

It follows that n was NOT the least element of T, contradicting the well-ordering Princip.

<u>Thm</u>: $\sqrt{2}$ is irrational   <u>NB</u>: $\sqrt{2} \approx 1.41$

<u>Pf</u> (via well-ordering)

If $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{N}$,

then $\sqrt{2}n$ is an natural for some n.

Let $T = \{n : \sqrt{2}n$ is a natural$\}$

If T is non-empty then it has a least element by the well-ordering principle. Let n be this least element.

Consider, $(\sqrt{2}-1)n = \underbrace{\sqrt{2}n}_{\text{natural}} - \underbrace{n}_{\text{natural}} \in \mathbb{N}$

<u>NB</u>: $\sqrt{2}n > n$ and so $\sqrt{2}n - n \in \mathbb{N}$.

We then have $(\sqrt{2}-1)n < n$ and

$$\sqrt{2}\left[(\sqrt{2}-1)n\right]$$

$$= (2-\sqrt{2})n = 2n - \sqrt{2}n \in \mathbb{N}$$

This contradicts the minimality of n.

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Strong Induction

Thm: The Strong Principle of Math Induction.
If S is a set of naturals such that:
(A) $1 \in S$
(B) $\{1,2,3,\dots,k\} \subseteq S \Rightarrow k+1 \in S$
then $S = \mathbb{N}$.

Ex: Every natural $n > 1$ is a product of one or more primes.

Pf: Let $S = \{n \in \mathbb{N} : n$ is a product of one or more primes $\}$

By definition, all primes are in S.
In particular $\{2,3\} \subseteq S$.

Suppose $\{2 \dots k\} \subseteq S$.
If $k+1$ is prime then $k+1 \in S$.
otherwise, is composite and

$$k+1 = a \cdot b$$

where $a, b < k$. Thus, $a \in S$ ad $b \in S$

Thus, a and b are products of primes.

Therefore $k+1$ is a product of primes.

Thm: Every natural number can be written as a sum of distinct powers of two

Ex: $1 = 2^0$ $\quad 4 = 2^2 \quad 7 = 2^2 + 2^1 + 2^0$

$2 = 2^1 \quad 5 = 2^2 + 2^0$

$3 = 2^1 + 2^0 \quad 6 = 2^2 + 2^1 \quad$ etc.

Pf: Let $S = \{ n \in \mathbb{N} : n$ can be written as a sum of distinct powers of two $\}$

By the example, $1, 2, 3, 4, 5, 6, 7 \in S$.

Suppose $\{1, 2, 3, \ldots, k\} \subseteq S$.

Consider $k+1$:

If $k+1$ is even then $k+1 = 2n$ for $n \leq k$.

By hypothesis n can be written as a sum of distinct powers of two. Thus, $k+1$ can be too.
(Multiplying by two preserves distinct.)

If $k+1$ is odd then k is even and can be written as a sum of distinct powers of two without 2^0.

Thus $k+1 = k + 2^0$ works.

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Recall,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \dots + \binom{n}{n}y^n$$

$\binom{n}{k}$ is the binomial coefficient "n choose k"

<u>Fact</u>: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ $\qquad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

<u>Pf</u>: $(x+y)^n$

$= (x+y)(x+y)^{n-1}$

$= (x+y)\left[\binom{n-1}{0}x^{n-1}y^0 + \binom{n-1}{1}x^{n-2}y^1 + \dots + \binom{n-1}{n-1}y^{n-1}\right]$

$= x[\dots] + y[\dots]$

Consider the term $x^{n-k}y^k$:

It occurs twice: $x \cdot \left[\binom{n-1}{k}x^{n-1-k}y^k\right]$

$+ y\left[\binom{n-1}{k-1}x^{n-1-(k-1)}y^{k-1}\right]$

$= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]x^{n-k}y^k$

<u>Thm</u>: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ is always a whole number.

<u>Pf</u>: $\binom{m}{0} = \binom{m}{m} = 1$ and all $\binom{n}{k}$ can be reduced to sums of this by applying the summation identity.

Thm : $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}=2^n$

Pf #1 (Induction)

Let $S=\{n : \binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}=2^n\}$

We have, $1\in S$ since $\binom{1}{0}+\binom{1}{1}=1+1=2=2^1$.

Suppose the claim holds for k, we get:

$\binom{k+1}{0}+\binom{k+1}{1}+\binom{k+1}{2}+\dots+\binom{k+1}{k+1}$

$=\binom{k+1}{0}+\underbrace{\binom{k}{0}+\binom{k}{1}}+\underbrace{\binom{k}{1}+\binom{k}{2}}+\dots+\underbrace{\binom{k}{k-1}+\binom{k}{k}}+\binom{k+1}{k+1}$

NB : Every term except $\binom{k}{0}$ and $\binom{k}{k}$ occurs twice

$=\binom{k+1}{0}+\binom{k}{0}+\binom{k}{k}+\binom{k+1}{k+1}+2\left[\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k-1}\right]$

$=2\binom{k}{0}+2\binom{k}{k}+2\left[\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k-1}\right]$

$=2\left[\binom{k}{0}+\binom{k}{1}+\dots+\binom{k}{k}\right]=2\cdot 2^k=2^{k+1}$.

Pf #2 (clever)

$2^n=(1+1)^n=\binom{n}{0}1^n1^0+\binom{n}{1}1^{n-1}1^1+\dots+\binom{n}{n}1^0 1^n$

$=\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}$