Constrained Optimization

Recall, the method of Lagrange Multiplier.

To optimize $f(x,y)$ subject to $g(x,y)=0$.

1. Define $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$
2. Solve $F_x = F_y = F_\lambda = 0$ to find crit. points.
3. Determine min/max.

Ex: minimize $f(x,y) = x\ln(x) + y\ln(y)$ subject to $x+y=1$.

Define $F(x,y,\lambda) = x\ln(x) + y\ln(y) - \lambda(x+y-1)$

$$\begin{cases} F_x = \ln(x) + 1 - \lambda = 0 \\ F_y = \ln(y) + 1 - \lambda = 0 \\ F_\lambda = x+y-1 = 0 \end{cases}$$

$\ln(x) + 1 - \lambda = 0$

$\ln(1-x) + 1 - \lambda = 0$

$\Rightarrow \ln(x) + 1 = \ln(1-x) + 1$

$\Rightarrow \ln(x) = \ln(1-x)$

$\Rightarrow x = 1-x \Rightarrow x = \frac{1}{2}$

$\ln(y) + 1 - \lambda = 0$

$\ln(1-y) + 1 - \lambda = 0$

$\Rightarrow \ln(y) = \ln(1-y)$

$\Rightarrow y = 1-y$

$\Rightarrow y = \frac{1}{2}$.

Ex: (winter 2014)

Your revenue is

$$f(x,y) = 20x + 26y - x^2 - 3y^2 + 9000$$

where the cost of x is \$2 and y is \$4.
If you have a budget of 280\$ maximize profit. (Assume the critical point is maximal).

Define $F(x,y,\lambda) = f(x,y) - \lambda(2x+4y-280)$

$$\begin{cases} F_x = 20 - 2x - 2\lambda = 0 \\ F_y = 26 - 6y + 4\lambda = 0 \\ F_\lambda = 2x + 4y - 280 = 0 \end{cases}$$

$\lambda = \frac{1}{2}(20 - 2x)$

$\lambda = \frac{1}{4}(26 - 6y)$

$\Rightarrow 10 - x = \frac{13}{2} - \frac{3}{2}y$

$\Rightarrow -x + \frac{3}{2}y = -\frac{7}{2}$

Solve:

$$\begin{cases} 2x + 4y = 280 \\ -x + \frac{3}{2}y = -\frac{7}{2} \end{cases}$$

$7y = 280 - 7$
$= 273$

$y = \frac{273}{7} = 39$

$x = -\frac{7}{2} + \frac{3}{2}(39)$
$= \frac{-7 + 117}{2} = 55.$

Thus,

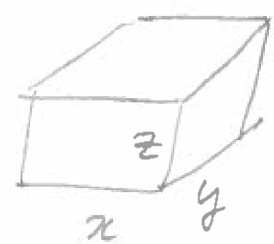
$$\begin{cases} x = 55 \\ y = 39 \end{cases}$$

Ex (Winter 2013)

A rectangular box will have volume $48 m^3$.
It will have no top.
If the bottom costs $4\$/m^2$
front/back $2\$/m^2$
left/right $1\$/m^2$

Find the cost minimizing dimensions.
(Assume the crit. point is a min.)

Let $f(x,y,z) = 2(2xz) + 1(2yz) + 4(xy)$
be the cost of materials



Let $g(x,y,z) = xyz - 48 = 0$
be the constraint.

Define $F(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y)$
$= 4xz + 2yz + 4xy - \lambda(xyz - 48)$

$$\begin{cases} F_x = 4z + 4y - \lambda(yz) = 0 \\ F_y = 2z + 4x - \lambda(xz) = 0 \\ F_z = 4x + 2y - \lambda(xy) = 0 \\ F_\lambda = xyz - 48 = 0 \end{cases}$$

Ex (con't):

we get: $\lambda = \dfrac{4z+4y}{yz} = \dfrac{4}{y} + \dfrac{4}{z}$

$\lambda = \dfrac{2z+4x}{xz} = \dfrac{2}{x} + \dfrac{4}{z}$

$\lambda = \dfrac{4x+2y}{xy} = \dfrac{4}{y} + \dfrac{2}{x}$

Thus, $\dfrac{4}{y} + \dfrac{4}{z} = \dfrac{2}{x} + \dfrac{4}{z} \implies \dfrac{4}{y} = \dfrac{2}{x}$

$\implies \dfrac{y}{4} = \dfrac{x}{2}$

$\implies y = 2x$

$\dfrac{2}{z} + \dfrac{4}{z} = \dfrac{4}{y} + \dfrac{2}{x} \implies \dfrac{4}{z} = \dfrac{4}{y}$

$\implies z = y$

Thus, we have:

$$xyz = 48 \implies x\cdot(2x)\cdot(2x) = 48$$

$$\implies x^3 = 12$$

$$\implies x = \sqrt[3]{12}.$$

Thus, $(x,y,z) = (\sqrt[3]{12}, 2\sqrt[3]{12}, 2\cdot\sqrt[3]{12})$

minimizes the cost.

Unconstrained Optimization

To optimize a function:

1. Find the critical points
2. Apply the second derivative test to determine min/max.

Ex: Find the minimum value of

$$f(x,y) = (x-1)^2 + (y-2)^2$$

Find critical points.

$f_x = 2(x-1) = 0$

$f_y = 2(y-2) = 0.$

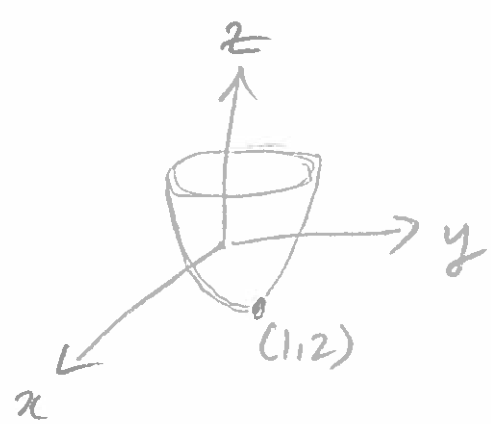
Thus, $(x,y) = (1,2)$ is the only critical point.

Apply the second derivative test.

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det(Hf) = 4 > 0$

$f_{xx} > 0$



Thus, $(x,y) = (1,2)$ is a minimum.

Ex (Winter 2013)

You manufacture two products A and B.

The selling price for A is $\$x$. It costs \$6 to make A.

B is $\$y$. \$20 B

The demand is $q_A(x,y) = 5(y-x)$

$$q_B(x,y) = 500 + 5(x-2y)$$

Find the selling prices that maximize profit.

We want to maxize:

$$F(x,y) = x\,q_A(x,y) + y\,q_B(x,y) - [6x + 20y]$$

$$= x[5(y-x)] + y[500 + 5(x-2y)]$$

$$- 6[x + 20y]$$

Find the critical points.

$$F_x = 5(y-x) - 5x + 5y - 6 = 10x - 10y - 6 = 0.$$

$$F_y = 5x + [500 + 5(x-2y)] + (-10)y - 120.$$

$$= 10x - 20y + 380 = 0.$$

Solve

$$\begin{cases} 10x - 10y = 6 \\ 10x - 20y = -380 \end{cases}$$

$$-10y = -386$$

Thus, $\begin{cases} y = \dfrac{386}{10} \\ x = \dfrac{392}{10}. \end{cases}$

Ex (winter 2012):

You manufacture a soft drink.

The cost to produce 100L is:

$$C(x,y) = 27x^3 - 72xy + 8y^2 + 2200$$

where x is kilos of sugar and y is kilos of flavour.

Minimize the production cost.

$$C_x = 81x^2 - 72y = 0 \Rightarrow y = \frac{81}{72}x^2$$

$$C_y = -72x + 16y = 0 \Rightarrow y = \frac{72}{16}x.$$

Thus, $\frac{81}{72}x^2 = \frac{72}{16}x$

$$\frac{81}{72}x^2 - \frac{72}{16}x = 0$$

$$x\left(\frac{81}{72}x - \frac{72}{16}\right) = 0$$

Thus $x = 0$ OR $x = \dfrac{\frac{72}{16}}{\frac{81}{72}} = \frac{72}{16} \cdot \frac{72}{81} = \frac{9 \cdot 8 \cdot 9 \cdot 8}{2 \cdot 8 \cdot 9 \cdot 9}$

$= \frac{8}{2} = 4.$

If $x=0$ then $y=0$.

If $x=4$ then $y = \frac{72}{16} = \frac{9}{2}$.

Ex (con't)

We have $C(0,0) = 2200$

$$C(4, \tfrac{9}{2}) = 27 \cdot 4^3 - 72 \cdot 4 \cdot (\tfrac{9}{2}) + 8 \cdot (\tfrac{9}{2})^2 + 2200$$

$$= 27 \cdot 4^3 - 72 \cdot 18 + 2 \cdot 81 + 2200$$

$$= 814.$$

Thus, we get: $C(4, \tfrac{9}{2}) < C(0,0)$.

It follow $C(4, \tfrac{9}{2})$ is the minimum.

We could also check using the second deriv test.

$C_{xx} = 161 \quad C_{xy} = -72$

$C_{yy} = 16 \quad C_{yx} = -72.$

Thus, we get: $HC = \begin{bmatrix} 161 & -72 \\ -72 & 16 \end{bmatrix}$

$$\det(HC) = 161 \cdot 16 - (-72)^2$$

$$= -2608.$$

Thus, we get a minimum.