Definition: matrix multiplication For $A = \big( a_{i\,j} \big) \in M_{n,k}(\mathbb{R})$, $B = \big( b_{i\,j} \big) \in M_{k,m}(\mathbb{R})$ we define $C = \big( c_{i\,j} \big) \in M_{n,m}(\mathbb{R})$ by $c_{i\,j} = \sum_{\ell=1}^{k} a_{i\,\ell}\, b_{\ell\, j} = \boldsymbol{a}_i \cdot \boldsymbol{b}_j$ where $\boldsymbol{a}_i$ is the $i^{th}$ row of $A$ and $\boldsymbol{b}_j$ is the $j^{th}$ column of $B$.

$C$ is called the **product** of $A$ and $B$ and we write $C = A\,B$.

Definition: determinant

(i) The determinant of a 1 by 1 matrix $\begin{pmatrix} a \end{pmatrix}$ is $a$.

(ii) Suppose a definition is provided for a $n-1$ by $n-1$ determinant.

Define

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \tilde{A}_{1j}$$

where $\tilde{A}_{ij}$ is the matrix obtained from $A$ by deleting the $i^{th}$ row and $j^{th}$ column.

notes:

The matrix $\tilde{A}_{ij}$ is sometimes called the $ij^{\text{th}}$ **minor matrix** of $A$.

Think of this approach as the definition by *expansion along the first row* or *expansion by minors along the first row.*

Properties of determinants

Let $A = \begin{pmatrix} \underline{v_1} \\ \underline{v_2} \\ \vdots \\ \underline{v_n} \end{pmatrix} \in M_n(\mathbb{R})$.

1. $\det A = \det A^t$

2. Interchanging two rows (or columns) multiplies the determinant by $(-1)$

For the rest of these properties, we will only mention rows — columns will follow because of property 1.

3. The determinant is linear in **each** row

(a) Multipying a row by $k$ multiplies the determinant by $k$.

Note that $\det\big(k\,A\big) = k^n \det A$

(b)
$$\det\begin{pmatrix} \underline{v_1} \\ \vdots \\ \underline{v_i} + \underline{v_i}' \\ \vdots \\ \underline{v_n} \end{pmatrix} = \det\begin{pmatrix} \underline{v_1} \\ \vdots \\ \underline{v_i} \\ \vdots \\ \underline{v_n} \end{pmatrix} + \det\begin{pmatrix} \underline{v_1} \\ \vdots \\ \underline{v_i}' \\ \vdots \\ \underline{v_n} \end{pmatrix}$$

Note that $\det\big(A + B\big) \neq \det\big(A\big) + \det\big(B\big)$

Note that the determinant is $n$–linear.

4. If two rows are proportional, the determinant is 0.

5. Adding a multiple of one row to another does not change the value of the determinant.

6. $\det\big(A\,B\big) = \big(\det A\big)\big(\det B\big)$.

Let $A = \left( a_{ij} \right) \in M_n(\mathbb{R})$. We define the **cofactor** of $a_{ij}$ denoted $c_{ij}$ by

$$c_{ij} = (-1)^{i+j} \det \tilde{A}_{ij} \ .$$

The matrix $C = \left( c_{ij} \right) \in M_n(\mathbb{R})$ is called the **cofactor matrix** of $A$.

$\Big($ Recall that $\tilde{A}_{ij}$ is the matrix obtained from $A$ by deleting the $i^{th}$ row and $j^{th}$ column and is sometimes called the $ij^{\text{th}}$ minor matrix of $A$. $\det \tilde{A}_{ij}$ can be called the $ij^{th}$ **minor** of $A$ or the **minor of the element** $a_{ij}$ of $A$.$\Big)$

In this context, a cofactor is sometimes called a signed minor.

<u>Note:</u> $c_{ij}$ is a scalar (real number) but $\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix.

We can now restate the definition of determinant in terms of cofactors.

(i) The determinant of a 1 by 1 matrix $\begin{pmatrix} a \end{pmatrix}$ is $a$.

(ii) Suppose a definition is provided for a $n-1$ by $n-1$ determinant.

Define

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} a_{1j}\, c_{1j} = a_{11}\, c_{11} + a_{12}\, c_{12} + \cdots + a_{1n}\, c_{1n}$$

where $c_{ij}$ is the cofactor of $a_{ij}$.

The transpose of the cofactor matrix $C$ of $A$ is called the **classical adjoint** of $A$ and denoted by adj $A$ ;i.e., adj $A = C^T$.

<u>Theorem:</u> If $A$ is any square matrix, then

$$A\,(\operatorname{adj} A) = (\det A)\, I = (\operatorname{adj} A)\, A\,.$$

In particular, if $\det A \neq 0$, the inverse of $A$ is given by

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A$$

where adj $A = C^T$, $C$ being the cofactor matrix of $A$.

| Windspeed (m/s) | Air Temperature (degrees C) | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **8** | **6** | **4** | **2** | **0** | **-2** | **-4** | **-6** | **-8** | **-10** | **-12** | **-14** | **-16** | **-18** | **-20** | **-22** | **-24** | **-26** | **-28** |
| **2** | 7 | 5 | 3 | 1 | -1 | -3 | -5 | -7 | -9 | -11 | -13 | -15 | -17 | -19 | -21 | -23 | -25 | -27 | -30 |
| **4** | 3 | 1 | -2 | -4 | -7 | -9 | -11 | -14 | -16 | -19 | -21 | -23 | -26 | -28 | -31 | -33 | -35 | -38 | -40 |
| **6** | 0 | -2 | -5 | -8 | -10 | -13 | -16 | -18 | -21 | -23 | -26 | -29 | -31 | -34 | -37 | -39 | -42 | -45 | -47 |
| **8** | -2 | -5 | -7 | -10 | -13 | -16 | -19 | -21 | -24 | -27 | -30 | -33 | -35 | -38 | -41 | -44 | -47 | -49 | -52 |
| **10** | -3 | -6 | -9 | -12 | -15 | -18 | -21 | -24 | -27 | -30 | -33 | -35 | -38 | -41 | -44 | -47 | -50 | -53 | -56 |
| **12** | -5 | -8 | -11 | -14 | -17 | -20 | -23 | -26 | -29 | -32 | -35 | -38 | -41 | -44 | -47 | -50 | -53 | -56 | -59 |
| **14** | -5 | -9 | -12 | -15 | -18 | -21 | -24 | -27 | -30 | -33 | -36 | -39 | -42 | -45 | -48 | -52 | -55 | -58 | -61 |
| **16** | -6 | -9 | -12 | -15 | -19 | -22 | -25 | -28 | -31 | -34 | -37 | -40 | -44 | -47 | -50 | -53 | -56 | -59 | -62 |
| **18** | -7 | -10 | -13 | -16 | -19 | -22 | -26 | -29 | -32 | -35 | -38 | -41 | -45 | -48 | -51 | -54 | -57 | -60 | -64 |
| **20** | -7 | -10 | -13 | -16 | -20 | -23 | -26 | -29 | -32 | -36 | -39 | -42 | -45 | -48 | -52 | -55 | -58 | -61 | -64 |
| **22** | -7 | -10 | -14 | -17 | -20 | -23 | -26 | -30 | -33 | -36 | -39 | -42 | -46 | -49 | -52 | -55 | -58 | -62 | -65 |
| **24** | -7 | -10 | -14 | -17 | -20 | -23 | -27 | -30 | -33 | -36 | -39 | -43 | -46 | -49 | -52 | -55 | -59 | -62 | -65 |
| **26** | -7 | -10 | -14 | -17 | -20 | -23 | -27 | -30 | -33 | -36 | -39 | -43 | -46 | -49 | -52 | -55 | -59 | -62 | -65 |