MAT B41 — Week 3a

Vector Geometry (§1.5)

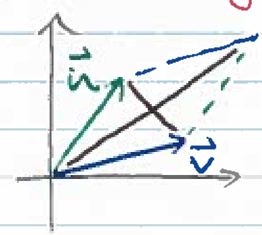
Recall from last week: $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$ and

$\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$ ← Length of $\vec{u}$.

We can use these ideas to prove theorems in geometry.

Thm: Suppose $\vec{u}$ and $\vec{v}$ span a parallelogram. If $\|\vec{u}\| = \|\vec{v}\|$ then the diagonals of the parallelogram form a right angle.

(!) Always draw the picture.



Observe that the diagonals are: $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

Pf: Assume $\|\vec{u}\| = \|\vec{v}\|$.
We calculate:

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{v}$$

$$= \underbrace{\|\vec{u}\|^2 - \|\vec{v}\|^2}_{\text{These are equal!}} = 0$$

Thus, $\vec{u}+\vec{v}$ is orthogonal to $\vec{u}-\vec{v}$.

Facts: $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

$\vec{a}\cdot(k\vec{b}) = k(\vec{a}\cdot\vec{b})$

Fact: (The triangle inequality) $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Limits

Def^n (p102): We write $\lim_{\vec{x} \to \vec{y}} f(\vec{x}) = \vec{z}$ if:

For all $\varepsilon > 0$ there is $\delta > 0$ such that:

$$\|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - \vec{z}\| < \varepsilon.$$

Ex (One dimensional) $\lim_{x \to 2} 5x = 10$

Given $\varepsilon > 0$ we pick $\delta < \frac{\varepsilon}{5}$.

?! We pick it this way so that the algebra works.

$|x-2| < \delta \implies 5|x-2| < 5\delta$

$\implies |5x - 10| < 5\delta < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$

Thus, $\lim_{x \to 2} 5x = 10$.

Ex (Two dimensional) $\lim_{(x,y) \to (1,2)} 2x + 3y = 8$.

Given $\varepsilon > 0$ we pick $\delta < \frac{\varepsilon}{K}$.

! We will decide the value of K later on.

$\|(x,y) - (1,2)\| < \delta$

$\implies |x-1| < \delta$ and $|y-2| < \delta$

$\implies -\delta < x - 1 < \delta$ and $-\delta < y - 2 < \delta$

$\implies -2\delta < 2x - 2 < 2\delta$ and $-3\delta < 3y - 6 < 3\delta$

$\implies -5\delta < 2x + 3y - 8 < 5\delta$

Thus, we pick $K = 5$ and get: $\|(x,y) - (1,2)\| < \delta \implies \|2x + 3y - 8\| < \varepsilon$.

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Ex Show $\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$.

(?) We always want to work with distances.

Observe: $\frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = \|(x,y)\|$

Thus, we are bounded above by $\|(x,y)\|$.

Given $\varepsilon > 0$ pick $\delta < \varepsilon$.

We get $\|(x,y)-(0,0)\| < \delta$

$$\Rightarrow \|(x,y)\| < \delta$$

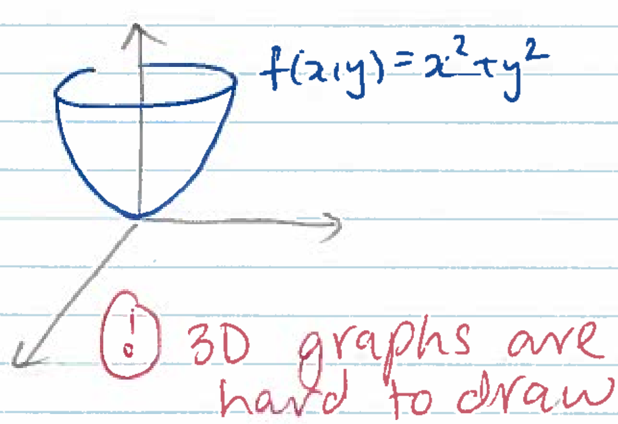
$$\Rightarrow \left| \frac{x^2}{\sqrt{x^2+y^2}} - 0 \right| = \frac{x^2}{\sqrt{x^2+y^2}} \leq \|(x,y)\| < \delta < \varepsilon.$$

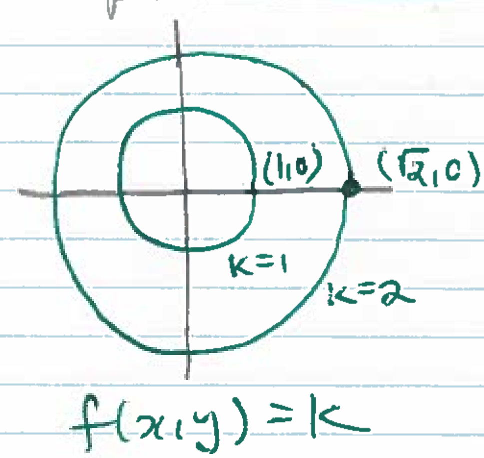
The only way to learn this technique is PRACTICE, PRACTICE, PRACTICE

Read over the examples in §2.2 and try them out.

Contour Plots

To understand a function $f: \mathbb{R}^2 \to \mathbb{R}$ we plot its "level curves" in the plane.





Limits and Continuity

Recall from last lecture the definition of limit:

For all $\varepsilon > 0$ there is $\delta > 0$:
$$\|\vec{x} - \vec{y}\| < \delta \Rightarrow \|f(\vec{x}) - \vec{z}\| < \varepsilon$$

Def^n: $f: \mathbb{R}^n \to \mathbb{R}^k$ is CONTINUOUS at $\vec{x} = \vec{y}$ if:
$$\lim_{\vec{x} \to \vec{y}} f(\vec{x}) = f(\vec{y})$$

Key Facts: If $f(\vec{x})$ and $g(\vec{x})$ are continuous at $\vec{x} = \vec{y}$ then:

The following are also continuous at $\vec{x} = \vec{y}$.

1. $kf(\vec{x})$
2. $f(\vec{x}) + g(\vec{x})$
3. $f(\vec{x}) \cdot g(\vec{x})$

Pf 1. If $\lim_{\vec{x} \to \vec{y}} f(\vec{x}) = f(\vec{y})$ then:

For all $\varepsilon_1 > 0$ there is $\delta_1 > 0$ such that
$$\|\vec{x} - \vec{y}\| < \delta_1 \Rightarrow \|f(\vec{x}) - f(\vec{y})\| < \varepsilon_1$$

Pf 1 (con't):

If $k=0$ then the theorem holds.

Assume $k \neq 0$.

Given $\varepsilon > 0$ we pick $\delta = \delta_1$ for $\varepsilon_1 = \frac{\varepsilon}{|k|}$.

We get:

$$\|\vec{x}-\vec{y}\| < \delta \Rightarrow \|f(\vec{x})-f(\vec{y})\| < \frac{\varepsilon}{|k|}$$

$$\Rightarrow |k|\,\|f(\vec{x})-f(\vec{y})\| < \varepsilon$$

$$\Rightarrow \|kf(\vec{x})-kf(\vec{y})\| < \varepsilon.$$

Pf 2: We need to show:

$$\lim_{\vec{x}\to\vec{y}} f(\vec{x})+g(\vec{x}) = f(\vec{y})+g(\vec{y}).$$

assuming $\boxed{\lim_{\vec{x}\to\vec{y}} f(\vec{x}) = f(\vec{y})}$ and $\boxed{\lim_{\vec{x}\to\vec{y}} g(\vec{x}) = g(\vec{y})}$

The key step:

$$\|f(\vec{x})+g(\vec{x}) - (f(\vec{y})+g(\vec{y}))\|$$

$$= \|f(\vec{x})-f(\vec{y}) + g(\vec{x})-g(\vec{y})\|$$

\# Triangle Inequality

$$\le \underbrace{\|f(\vec{x})-f(\vec{y})\|}_{<\varepsilon/2} + \underbrace{\|g(\vec{x})-g(\vec{y})\|}_{<\varepsilon/2} < \varepsilon.$$

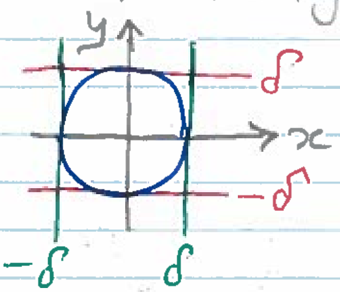
The Circle - Box Argument

Ex: Show that $\lim\limits_{(x,y)\to(0,0)} \min\{2|x|, 4|y|\} = 0$

Given $\varepsilon > 0$ we need $\delta > 0$ such that:

$$\|(x,y)-(0,0)\| < \delta \Longrightarrow \|\min\{2|x|,4|y|\} - 0\| < \varepsilon.$$

Observe: If $\|(x,y)\| < \delta$ then $|x| < \delta$ and $|y| < \delta$.



The ball represents: $\|(x,y)\| < \delta$

The box represents: $|x| < \delta$ and $|y| < \delta$

We want: $2|x| < \varepsilon$ and $4|y| < \varepsilon$.

Thus, we pick: $\delta < \frac{\varepsilon}{4}$

We obtain:

$$\begin{aligned} \|(x,y)-(0,0)\| < \delta &\Rightarrow |x| < \delta \text{ and } |y| < \delta \\ &\Rightarrow 2|x| < 2\delta \text{ and } 4|y| < 4\delta \\ &\Rightarrow 2|x| < 2\left(\tfrac{\varepsilon}{4}\right) \text{ and } 4|y| < 4\left(\tfrac{\varepsilon}{4}\right) \end{aligned}$$

Thus we obtain:

$$\|(x,y)-(0,0)\| < \delta \Longrightarrow \|\min\{2|x|,4|y|\} - 0\| < \varepsilon.$$

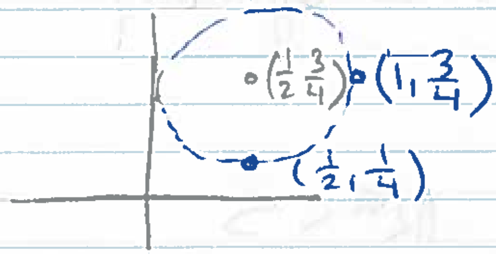
MAT 341 — Week 3b

The most important use of geometry:

Def^n: The OPEN DISK of radius $r > 0$ at $\vec{x}_0$ is:

$$D_r(\vec{x}_0) = \{\vec{x} : \|\vec{x} - \vec{x}_0\| < r\}$$

Ex: Sketch $D_{\frac{1}{2}}\left(\left(\frac{1}{2}, \frac{3}{4}\right)\right)$ in the plane.



? What is $D_r(x) \subseteq \mathbb{R}$?

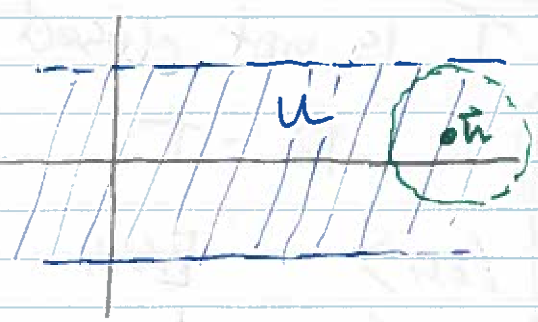
The open interval $(x-r, x+r)$.

Def^n: A set $U \subset \mathbb{R}^n$ is OPEN if:
For all $\vec{u} \in U$ there is $\varepsilon > 0$ such that:

$$D_\varepsilon(\vec{u}) \subset U.$$

Ex: Check that the set $U = \{(x, y) : |y| < 2\}$ is open in the plane.

! Sketch the region.



For $\vec{u} = (x, y) \in U$ we get:

$|y| < 2$

Thus, we pick $\varepsilon = 2 - |y| > 0$

This gives a ball around $\vec{u}$.

Discuss: Check that $D_r(\vec{x})$ is open.

For $\vec{a} \in D_r(\vec{x})$ pick $\varepsilon = r - \|\vec{a} - \vec{x}\| > 0$.

MAT B41 — Week 3c

Derivatives

Defn: For a function $f: \mathbb{R}^2 \to \mathbb{R}$ we define

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

These are the PARTIAL DERIVATIVES of $f(x,y)$.

Ex: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = x^2y + y^3$.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2y + y^3] - [x^2y + y^3]}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2)y + y^3 - (x^2y + y^3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2y} + 2xyh + h^2y + \cancel{y^3} - \cancel{x^2y} - \cancel{y^3}}{h}$$

$$= \lim_{h \to 0} \frac{h(2xy + hy)}{h} = \lim_{h \to 0} 2xy + hy = 2xy$$

Similarly, $\frac{\partial f}{\partial y} = x^2 + 3y^2$

Defn: For a function $f: \mathbb{R}^n \to \mathbb{R}$ we have
PARTIAL DERIVATIVES

$$\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{e}_i) - f(\vec{x})}{t}$$

where $\vec{e}_i = (0\ 0 \ldots 1 \ldots 0)$
↑ i^{th} position.

Ex: Find $\frac{\partial f}{\partial z}$ if $f(x,y,z) = xyz^2$.

$$\frac{\partial f}{\partial z} = \lim_{t \to 0} \frac{f(x,y,z+t) - f(x,y,z)}{t}$$

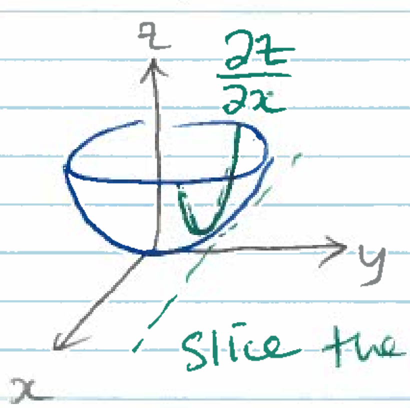
$$= \lim_{t \to 0} \frac{xy(z+t)^2 - xyz^2}{t}$$

$$= \lim_{t \to 0} \frac{xy(\cancel{z^2} + 2zt + t^2) - \cancel{xyz^2}}{t}$$

$$= \lim_{t \to 0} \frac{2xyzt + xyt^2}{t}$$

$$= \lim_{t \to 0} 2xyz + xyt = 2xyz.$$

In general, to compute $\frac{\partial f}{\partial x_i}$ assume that all the variables other than x_i are constant and differentiate in the variable x_i.



Slice the graph along a plane