MAT B41 — Week 5b

Taylor Series

Σ notation review?

In one variable we have the following:

$$g(x_0+h) = \lim_{n\to\infty} \sum_{k=0} \frac{1}{k!} f^{(k)}(x_0)\, h^k$$

where $k! = k(k-1)\ldots 3\cdot 2\cdot 1$ and
$f^{(k)} = k^{th}$ derivative of $f(x)$.

Def^n: The n^{th} ORDER TAYLOR POLYNOMIAL of $f(x)$ at x_0 is:

$$g(x_0+h) \approx \frac{1}{0!} f^{(0)}(x_0)\, h + \ldots + \frac{1}{n!} f^{(n)}(x_0) h^n$$

Ex: Find the second order Taylor polynomial of $f(x) = e^{2x}$ at $x_0 = 0$.

Check with Desmos!

\# Calculate $f^{(n)}(0)$

$$f(x) = e^{2x} \Rightarrow f^{(n)}(x) = 2^n e^x \Rightarrow f^{(n)}(0) = 2^n$$

\# Apply Taylor's formula

$$g(0+h) = \frac{1}{0!} \cdot f^{(0)}(0) + \frac{1}{1!} f^{(1)}(0)\, h$$

$$+ \frac{1}{2!} f^{(2)}(0)\, h^2$$

$$= 1 + 2h + 4h^2$$

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Question: Why do we define Taylor series in this way?

We want to approximate functions by polynomials. Taylor series are defined so that polynomials approximate themselves.

Fact: If $f(x)$ is a polynomial and $g(x)$ is its Taylor series then $g(x) = f(x)$.

Ex: Find the Taylor series at $x_0 = 0$ of $f(x) = 1 + 3x + x^2$.

Calculate $f^{(n)}(0)$.

$f^{(0)}(x) = f(x) = 1 + 3x + x^2 \Rightarrow f^{(0)}(0) = 1$

$f^{(1)}(x) = 3 + 2x \Rightarrow f^{(1)}(0) = 3$

$f^{(2)}(x) = 2$ ⊛ $\Rightarrow f^{(2)}(0) = 2$

$f^{(3)}(x) = 0 \Rightarrow f^{(n)}(0) = 0 \ (n \geq 3)$

Apply Taylor's formula

$$g(0+h) = \frac{1}{0!} f^{(0)}(0) + \frac{1}{1!} f^{(1)}(0) h + \underbrace{\frac{1}{2!} f^{(2)}(0) h^2}_{⊛}$$

$$= 1 + 3h + \frac{1}{2!} 2 \cdot h^2 = 1 + 3h + h^2$$

? The factorial ensures derivatives cancel out.

$f(x) = x^n \Rightarrow f^{(n)}(0) = n!$

<u>Discuss</u>: What about $x_0 \neq 0$?
check the Taylor for $f(x) = x^2 + x + 1$
at $x_0 = 2$. It should give $g(x) = f(x)$.

<u>Trig Functions</u>

<u>Ex</u>: Find the Taylor series of $f(x) = \sin(x)$.

Calculate $f^{(n)}(x)$

$$
\begin{aligned}
f^{(0)}(x) &= f(x) = \sin(x) \Rightarrow f^{(0)}(0) = 0 \\
f^{(1)}(x) &= \cos(x) \qquad f^{(1)}(0) = 1 \\
f^{(2)}(x) &= -\sin(x) \qquad f^{(2)}(0) = 0 \\
f^{(3)}(x) &= -\cos(x) \qquad f^{(3)}(0) = -1 \\
f^{(4)}(x) &= \sin(x) \qquad f^{(4)}(0) = 0
\end{aligned}
$$

Repeats! $\left\{ \begin{aligned} f^{(5)}(x) &= \cos(x) \\ f^{(6)}(x) &= -\sin(x) \\ f^{(7)}(x) &= -\cos(x) \\ f^{(8)}(x) &= \sin(x) \end{aligned} \right.$

Clever observation: Only odd terms are non-zero!

We get: $f^{(2k+1)}(0) = (-1)^k$

Thus,

$$\sin(h) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} h^{2k+1}$$

<u>Discuss</u>: Repeat this analysis for $\cos(h)$.

$$\cos(h) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} h^{2k}$$

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The Most Beautiful Theorem

Theorem (Euler) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
(i is defined so that $i^2 = -1$)

From last lecture, $e^h = \sum_{k=0}^{\infty} \frac{h^k}{k!}$

We obtain:

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k \theta^k}{k!}$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!}}_{k \text{ even}} + \underbrace{\sum_{n=0} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!}}_{k \text{ odd}}$$

$$= \sum_{n=0}^{\infty} \frac{(i^2)^n \theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(i^2)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i\sum_{n=0} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \cos(\theta) + i\sin(\theta)$$

? Take $\theta = \pi$ and get: $e^{i\pi} = -1 + i\cdot 0$
$\Leftrightarrow \boxed{e^{i\pi} + 1 = 0}$