MATB41 LEC 8 (1 hr) July 3, 2018

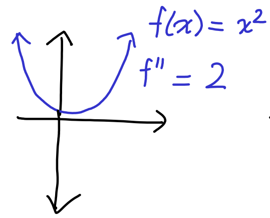
Second Derivative Test

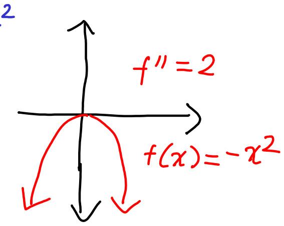
Defn: We say $\vec{x} = \vec{c}$ is a CRITICAL POINT of $f(\vec{x})$ if f is not diff at $\vec{x} = \vec{c}$ OR $\nabla f(\vec{c}) = 0$

Alternatively $\frac{\partial f}{\partial x_i}(\vec{c}) = \frac{\partial f}{\partial y}(\vec{c}) = 0$

In one variable, $f''(c) < 0 \Rightarrow c$ is a max

$f''(c) > 0 \Rightarrow c$ is a min





How does this work in multiple variables?

Defn: The second deriv. of $f: \mathbb{R}^n \to \mathbb{R}$ is the HESSIAN MATRIX

$$\boxed{Hf = \left[f_{x_i x_j} \right]}$$

Ex: Find the Hessian of $f(x,y) = x^2 + y^2$

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad f_x = 2x \quad f_y = 2y$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Ex: Find the Hessian of $f(x,y) = -x^2 - y^2$

$$Hf = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Ex: Find the Hessian of $f(x,y) = xe^y$ at the point $(x,y) = (2,0)$

get 1st deriv

$f_x = e^y \qquad f_y = xe^y$

get 2nd deriv

$$Hf = \begin{bmatrix} 0 & e^y \\ e^y & xe^y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

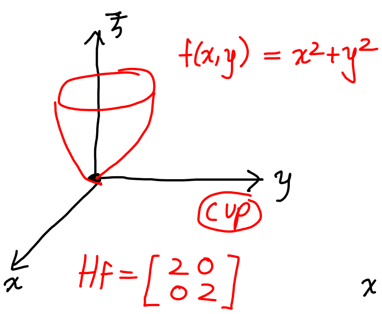
Defn: We say $\vec{x} = \vec{c}$ is a LOCAL MAX if $f(\vec{c}) \geq f(\vec{y})$ $\forall \vec{y}$ near $\vec{c}$

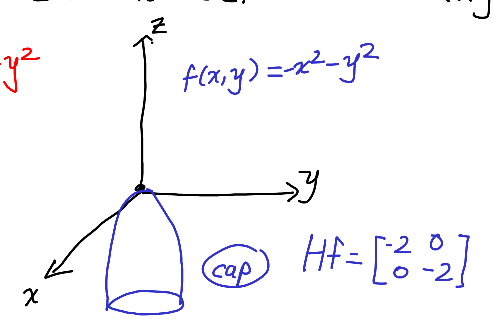
We say $\vec{x} = \vec{c}$ is a LOCAL MIN if $f(\vec{c}) \leq f(\vec{y})$ $\forall \vec{y}$ near $\vec{c}$

Fact: If $f(\vec{x})$ is diff^nble near $\vec{x} = \vec{c}$ and $\vec{x} = \vec{c}$ is a local min/max then $\nabla f(\vec{c}) = 0$

(Alternatively $\vec{x} = \vec{c}$ is a crit point)

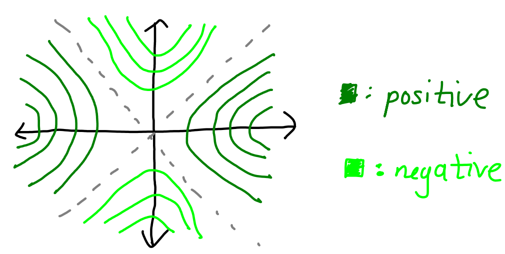
If $\nabla f \neq 0$, then we can increase/decrease locally





Ex: Find Hf for $f(x,y) = x^2 - y^2$

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{SADDLE}$$



Fact: Given $f: \mathbb{R}^2 \to \mathbb{R}$ diff^nble near $\vec{x} = \vec{c}$, a crit pt

We want to know "Is $\vec{x} = \vec{c}$ a max/min/saddle"

If $\det(Hf) < 0$ we get a saddle
If $\det(Hf) > 0$ and $f_{xx} > 0$ we get min
If $\det(Hf) > 0$ and $f_{xx} < 0$ we get max

$\Longrightarrow$

If $\det(Hf) = 0$ then no conclusion

Ex: Find and classify crit pts of $f(x,y) = x^2 + 2xy + y^2$

solve $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x} = 2x + 2y = 0 \quad \frac{\partial f}{\partial y} = 2x + 2y = 0$ When $\boxed{y = -x}$

classify using 2nd deriv. test

$$H_f = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \det(H_f) = 0 \quad \therefore \text{Inconclusive}$$

We have $f(x,y) = x^2 + 2xy + y^2 = (x+y)^2 \geq 0$
The crit pts are all minima

MATB41 LEC 8 (2hr) July 5, 2018

Mock Final : Fri July 27, SY110 @ 12:00pm

The Second Derivative Test ($F:\mathbb{R}^2 \to \mathbb{R}$)

Suppose $\vec{x}=\vec{c}$ is a critical pt and $f(\vec{x})$ is diffble near $\vec{x}=\vec{c}$

If $\det(H_f)>0$ and $f_{xx}>0$ then $\vec{x}=\vec{c}$ is a minimum

CUP $\to f(x,y)=x^2+y^2$

If $\det(H_f)>0$ and $f_{xx}<0$ then $\vec{x}=\vec{c}$ is a maximum

CAP $\to f(x,y)=-x^2-y^2$

Ex: Classify the critical point of $f(x,y)=xy+\frac{1}{x}+\frac{1}{y}$ at $(x,y)=(1,1)$

calculate H_f

$f_x = y-\frac{1}{x^2}$ $\quad f_y = x-\frac{1}{y^2}$ $\quad H_f=\begin{bmatrix}\frac{2}{x^3} & 1\\ 1 & \frac{2}{y^3}\end{bmatrix}$

apply 2nd deriv test

$H_f(1,1)=\begin{bmatrix}2&1\\1&2\end{bmatrix}$ $\quad \det(H_f)=3>0$ $\quad f_{xx}=2>0$

Thus, $(x,y)=(1,1)$ is a minimum.

Challenging Exercise : Classify Crit pts of Ex 2

Ex: Find the crit pts of: $f(x,y)=(x^2+3y^2)e^{1-x^2-y^2}$

take the partial derivs

$\frac{\partial f}{\partial x} = 2xe^{1-x^2-y^2}+(x^2+3y^2)\cdot e^{1-x^2-y^2}\cdot(-2x)$

$= 2x(1-x^2-3y^2)\cdot e^{1-x^2-y^2}$ # factoring

$\frac{\partial f}{\partial y} = 6y\cdot e^{1-x^2-y^2}-(x^2+3y^2)\cdot e^{1-x^2-y^2}(-2y)$

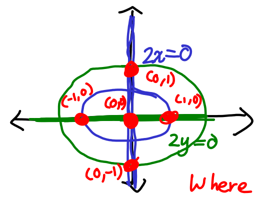
$= 2y(3-x^2-3y^2)\cdot e^{1-x^2-y^2}$

One of these terms have to be 0

The e term cannot

$0=1-x^2-3y^2 \Leftrightarrow x^2+3y^2=1$

This is an ellipse slightly skinnier in y direction



Where green and blue line touch are crit pts

$0=3-0^2-3y^2 \Leftrightarrow y=\pm1$ $\quad x=\pm1 \Leftrightarrow x^2+0^2=1$

The Multivariable Second Deriv Test

Let $\vec{x}=\vec{c}$ be a crit pt of f

$H=H_f(\vec{c})$ be the Hessian at $\vec{x}=\vec{c}$

H_k = "upper left $k\times k$ corner of H"

i.e.: $H=\begin{bmatrix}1&2&3\\2&2&3\\3&3&3\end{bmatrix}$ $\quad H_1=[1]$ $\quad H_2=\begin{bmatrix}1&2\\2&2\end{bmatrix}$ $\quad H_3=H$

Fact : If $\det(H_1)>0$ and $\det(H_2)>0 \dots$ $\det(H_k)>0$ $\forall k$ then $\vec{x}=\vec{c}$ is a minimum

If $\det(H_1)<0$ and $\det(H_2)>0$ and $\det(H_3)<0 \dots$ k odd $\Rightarrow -ve$; k even $\Rightarrow +ve$ then $\vec{x}=\vec{c}$ is a max

Ex: Generalized Cup and Cap

$f(x_1,x_2,\dots x_n)=x_1^2+x_2^2+\dots+x_n^2$

$H_f=\begin{bmatrix}2&0&0\\0&2&0\\0&0&\ddots\,2\end{bmatrix} \Rightarrow \det(H_k)=2^k$ cup

$g(x_1,x_2,\dots x_n)=-x_1^2-x_2^2-\dots-x_n^2$

$H_g=\begin{bmatrix}-2&0&0\\0&-2&0\\0&0&\ddots\,-2\end{bmatrix} \Rightarrow \det(H_k)=(-2)^k$ cap

Ex: Find and classify crit pts of $f(x,y,z)=x^2+xy+y^2+z^2$

find the crit pts

$\underbrace{\frac{\partial f}{\partial x}=2x+y=0 \quad \frac{\partial f}{\partial y}=x+2y=0 \quad \frac{\partial f}{\partial z}=2z=0}$

$(x,y,z)=(0,0,0)$

classify the crit pt

$H_f=\begin{bmatrix}2&1&0\\1&2&0\\0&0&2\end{bmatrix}$ $\quad \det(H_1)=2$ $\quad \det(H_2)=\begin{vmatrix}2&1\\1&2\end{vmatrix}=3$

$\det(H_3)=6$ All >0

Thus $(x,y,z)=(0,0,0)$ is min

Optimization

Ex: Find the pt on the graph $z=2x+3y$ closest to $(1,2,3)$

Introduce a Function

$[d(x,y)]^2=(1-x)^2+(2-y)^2+(3-\overbrace{2x-3y}^{z})^2$

(!) If d^2 is minimized then d is minimized

Find the crit pts

$\frac{\partial f}{\partial x}=-2(1-x)\quad 2(3-2x-3y)(-2)$

$=-(2-2x)-12+8x+12y$

$=10x+12y-14$

$\frac{\partial f}{\partial y}=2(2-y)(-1)+2(3-2x-3y)(-3)$

$=-4+2y-18+12x+18y$

$=20y+12x-22$

$\begin{cases}10x+12y-14=0\\12x+20y-22=0\end{cases} \Leftrightarrow \begin{cases}5x+6y-7=0\\6x+10y-11=0\end{cases}$

solve for (x,y)

$(x,y)=(\frac{2}{7},\frac{13}{14})$

$z=\frac{47}{14}$

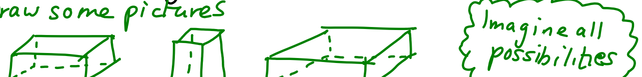
Find the Hessian

$H_f=\begin{bmatrix}10&12\\12&20\end{bmatrix}$ $\quad \det(H_f)=200-144>0$

Thus, $(x,y,z)=\left(\frac{2}{7},\frac{13}{14},\frac{47}{14}\right)$ is the closest pt to $(1,2,3)$

Ex: Find the rectangular prism of volume V having the least surface area.

Draw some pictures



Imagine all possibilities

$SA(x,y,z)=2xy+2yz+2xz \Rightarrow SA(x,y)=2xy+2y\left(\frac{V}{xy}\right)+2x\left(\frac{V}{xy}\right)$

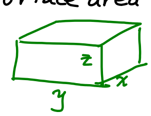
Observe : $z=\frac{V}{xy}$

$=\boxed{2\left(xy+\frac{V}{x}+\frac{V}{y}\right)}$

State mathematically

Minimize $SA(x,y,z)$ the surface area subject to the constraint $V=xyz$

use the constraints to reduce the number of vars



find crit pts

$\frac{\partial SA}{\partial x}=2\left(y-\frac{V}{x^2}\right)=0$ $\quad \frac{\partial SA}{\partial y}=2\left(x-\frac{V}{y^2}\right)=0$

Thus, $y=\frac{V}{x^2}$ and $x=\frac{V}{y^2}$

$y=\frac{V}{x^2}=\frac{V}{\left(\frac{V}{y^2}\right)^2} \Leftrightarrow y=\frac{V}{\frac{V^2}{y^4}} \Leftrightarrow \boxed{Vy=y^4}$

$Vy-y^4=0$

$y(V-y^3)=0$

$\cancel{y=0}$ $\quad \boxed{y=\sqrt[3]{V}}$

Thus $y=\sqrt[3]{V}$; $x=\frac{V}{(\sqrt[3]{V})^2}=\sqrt[3]{V}$

$z=\frac{V}{\sqrt[3]{V}\cdot\sqrt[3]{V}}=\sqrt[3]{V}$

Thus cube has lowest SA w/ Volume V

MATB41 LEC 8 (2hr Cont)

#Check this is a min.

$$HSA = \begin{bmatrix} \frac{4V}{x^3} & 2 \\ 2 & \frac{4V}{y^3} \end{bmatrix} \qquad HSA = (\sqrt[3]{V}, \sqrt[3]{V}) = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\det(HSA) = 16 - 4 > 0$$

Ex: (§3.3 Q31)

Write 120 as a sum of three numbers s.t the sum of the products taken 2 at a time is maximal.

#Write mathematically

Maximize $P(x,y,z) = xy + yz + xz$

subject to $x + y + z = 120$

#use constraint to reduce variables

$z = 120 - x - y$

$$\begin{aligned} P(x,y) &= xy + y(120 - x - y) + x(120 - x - y) \\ &= xy + 120x - x^2 - xy + 120y - xy - y^2 \\ &= 120x + 120y - xy - x^2 - y^2 \end{aligned}$$

#find crit pts

$\frac{\partial f}{\partial x} = 120 - y - 2x = 0 \iff 120 = 2x + y$

$\frac{\partial f}{\partial y} = 120 - x - 2y = 0 \iff 120 = x + 2y$

$(x, y) = (40, 40)$

#classify crit pts

$$H_P = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

$\det(H_P) = 3 > 0$

and $P_{xx} = -2 < 0$

Thus $(x,y,z) = (40,40,40)$ is maximal.

Problem Solving Heuristic

There are values of SA much larger than the cube, thus the crit pt <u>should</u> be a minimum

<u>Summary</u>

→ Critical Points
- f not diff<u>able</u>
- $\nabla f = 0$

→ Hessian matrix H_f

→ H_k the $k \times k$ submatrix

→ Second Deriv Test

$f: \mathbb{R}^n \to \mathbb{R}$ and $f: \mathbb{R}^2 \to \mathbb{R}$

→ Constraints to reduce vars