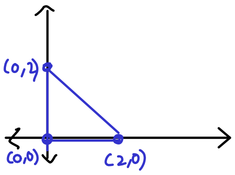
MATB41 LEC 9 (1 hr) July 10, 2018

Constrained Optimization

So far: "Minimize/maximize $f(\vec{x})$ on its domain."

We get interesting behaviour when we restrict the domain.

Ex: Maximize $f(x,y) = x^2+y^2$ on the triangle w/ vertices $(0,0)$ $(2,0)$ and $(0,2)$



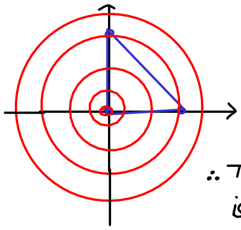
\# Find the critical points

$\begin{cases} f_x = 2x = 0 \\ f_y = 2y = 0 \end{cases} \Rightarrow (x,y) = (0,0)$

\# Apply 2nd deriv test

$H_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow$ minimum

\# check the boundary ("end points")



Look for the largest level set that still touches the boundary

We get $f(0,2) = f(2,0) = 4$

$\therefore$ The maximum of $f(x,y)$ on the triangle is $f(0,2) = f(2,0) = 4$

Ex: (Final 2018)

Find the maximum of $f(x,y,z) = x^2 - 3x + y^2 - 2y + z^2 - 4z + 1$ on the ball $x^2+y^2+z^2 \le 9$ ← Bounded & Closed $\therefore$ Has max/min by EVT

\# Find crit pts

$\left.\begin{aligned} f_x &= 2x-3 = 0 \\ f_y &= 2y-2 = 0 \\ f_z &= 2z-4 = 0 \end{aligned}\right\} \Rightarrow (x,y,z) = (\tfrac{3}{2}, 1, 2)$

This pt is in the ball (squares are ≤ 9)

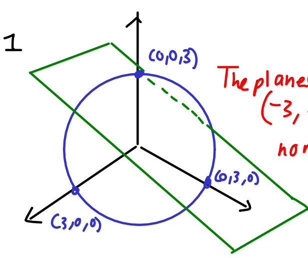
\# Apply 2nd deriv test

$H_f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ Critical point is a minimum ↗ but we wanted max $\therefore$ use other methods

$f(x,y,z) = x^2 - 3x + y^2 - 2y + z^2 - 4z + 1$

\# Use the constraint $x^2+y^2+z^2 = 9$

$f(x,y,z) = 9 - 3x - 2y - 4z + 1$
$= 10 - 3x - 2y - 4z$



The planes all have $(-3,-2,-4)$ normal

Exercise:

Find where the line $l: (-3,-2,-4)t$ meets w/ the sphere $x^2+y^2+z^2 = 9$

Ex: Maximize $f(x) = x^2$ on $[1,2]$

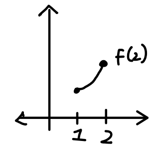
\# Find the critical points

$f'(x) = 2x = 0 \Rightarrow x = 0$

(!) 0 is outside of $[1,2]$

\# Check the end points

$f(1) = 1 \quad f(2) = 4$

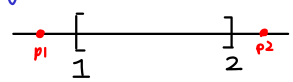


Def^n: $S \subseteq \mathbb{R}^n$ is OPEN if:
$\forall x \in S,\ \exists \varepsilon > 0$ s.t $D_\varepsilon(x) \subseteq S$

"Any pt in the Set can be surrounded by a disk"

Def^n: A set $T \subseteq \mathbb{R}^n$ is CLOSED if:
$S = \mathbb{R}^n \setminus T$ is open

"Every point outside of T can be surrounded by a disk"



p_1, p_2 can be surrounded by disk

Def^n: A set $U \subseteq \mathbb{R}^n$ is BOUNDED if there is N s.t $U \subseteq D_N(\vec{0})$

"U can be surrounded by a disk"

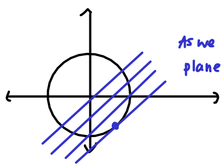
> Thm: (Extreme Value)
> If S is closed and bounded then any continuous $f: S \to \mathbb{R}$ achieves a maximum and minimum
> $f(x_0) = M \qquad f(x_1) = m$

Why closed and bounded?

Closed \ Bounded	YES	NO
YES	✓	$[0,\infty)$ $f(x) = \frac{1}{1+x^2}$ max but no min
NO	$(0,1)$ $f(x) = x$ no max or min	$(0,\infty)$ $f(x) = \frac{1}{1+x^2}$ no max or min

Optimization

Ex: Maximize $f(x,y,z) = 10 - 3x - 2y - 4z$ (From prev. lec)
on the sphere $x^2+y^2+z^2 = 9$ ← level curves



As we increase k, plane shifts down

The maximum should be tangent to the constraint

\# Find the point of tangency

All the planes $f(x,y,z)=k$ have the normal $\vec{n} = (-3,-2,-4)$

\# Find where $l: (-3,-2,-4)$ meets the sphere

$(-3t)^2 + (-2t)^2 + (-4t)^2 = 9 \iff 9t^2 + 4t^2 + 16t^2 = 9$
$\iff 29t^2 = 9$
$\iff t = \pm \frac{3}{\sqrt{29}}$

We pick $(x,y,z) = \frac{3}{\sqrt{29}}(-3,-2,-4)$ as the maximum. (The -ve version would give min)

The Method of Lagrange Multipliers

To optimize $f(\vec{x})$ subject to $g(\vec{x}) = 0$

1. Define $\mathcal{L}(\vec{x},\lambda) = f(\vec{x}) - \lambda g(\vec{x})$
2. Find the crit pts of $\mathcal{L}$
3. Select the crit pts that min/maximizes $f(\vec{x})$

Observations

① $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0$ ② $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$

$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0$ $\iff -g(\vec{x}) = 0$ $\iff g(\vec{x}) = 0$

$\iff \begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \end{cases}$

$\iff \nabla f = \lambda \nabla g$ ← Parallel gradients

Ex: Maximize $f(x,y,z) = \ln(x) + \ln(y) + \ln(z)$ subject to $x+y+z = 1$

\# Rewrite constraint

$g(x,y,z) = x+y+z-1 = 0$

\# Setup Lagrange

$\mathcal{L} = [\ln(x) + \ln(y) + \ln(z)] - \lambda(x+y+z-1)$

\# Find the crit pts

$\begin{cases} \mathcal{L}_x = \frac{1}{x} - \lambda = 0 \iff \frac{1}{x} = \lambda \\ \mathcal{L}_y = \frac{1}{y} - \lambda = 0 \iff \frac{1}{y} = \lambda \\ \mathcal{L}_z = \frac{1}{z} - \lambda = 0 \iff \frac{1}{z} = \lambda \\ \mathcal{L}_\lambda = x+y+z-1 = 0 \iff x+y+z = 1 \end{cases}$

$\Rightarrow \frac{1}{x} = \frac{1}{y} = \frac{1}{z} = y$

$\Rightarrow x = y = z$

We solve: $x+y+z = 3x = 1$

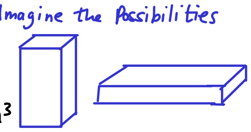
We obtain $(x,y,z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ∴ the crit pt is a max

You can make $f(x,y,z)$ -ve $y = 0.01$ $z = e^{-1000}$

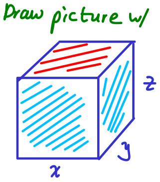
Ex: Suppose you make boxes
- The top and bottom cost $2^{\$}/m^2$
- sides $3^{\$}/m^2$

Design the least expensive box w/ Volume $1m^3$

Imagine the Possibilities



\# Draw picture w/ variables



\# state the goal

minimize $f(x,y,z) = \text{cost} = 2(2xy) + 3(2xz) + 3(2yz)$
subject to $1 = V = xyz$

\# Setup Lagrange

$\mathcal{L}(x,y,z,\lambda) = [4xy + 6xz + 6yz] - \lambda(xyz - 1)$

\# Find crit pts ① Tricky move ↙ $xyz = 1$

$\mathcal{L}_x = 4y + 6z - \lambda yz = 0 \Rightarrow 4xy + 6xz - \lambda xyz = 0$
$\mathcal{L}_y = 4x + 6z - \lambda xz = 0 \Rightarrow 4xy + 6yz - \lambda xyz = 0$
$\mathcal{L}_z = 6x + 6y - \lambda xy = 0 \Rightarrow 6xz + 6yz - \lambda xyz = 0$
$\mathcal{L}_\lambda = xyz - 1$

We get:

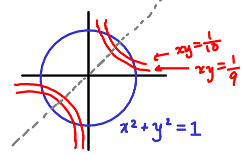
$\cancel{4xy} + 6xz = \lambda = \cancel{4xy} + 6yz$
$6xz = 6yz \Rightarrow$ Either $\boxed{\cancel{z=0}}$ No Volume OR $\boxed{x = y}$

$4xy + \cancel{6yz} = \lambda = 6xz + \cancel{6yz}$
$4xy = 6xz \Rightarrow \boxed{\cancel{x=0}}$ OR $\boxed{y = \frac{3}{2}z}$

We get $x = y = \frac{3}{2}z$

Exercise: solve for (x,y,z)

Ex: Find the maximum of $f(x,y) = xy$ on $x^2+y^2 = 1$



\# draw the constraint
\# draw the level set

We need to find the pt of tangency
Observe: both curves are symmetric about $y = x$
Assume $\boxed{y = x}$

$x^2+y^2 = 1 \Rightarrow x^2+x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow \boxed{x = \pm\frac{1}{\sqrt{2}}}$

We obtain:
$f(x,y) = xy$ is tangent to $x^2+y^2 = 1$ at $(x,y) = (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$
We get $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$, $f(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{1}{2}$ → seem to be the maxima

Look at the gradients:	At the P.O.I:
$f(x,y) = xy \Rightarrow \nabla f = (y,x)$	these are parallel!
$g(x,y) = x^2+y^2 \Rightarrow \nabla g = (2x, 2y)$	⇒ Same tangent plane

Ex: Optimize $f(x,y) = xy$ on $x^2+y^2 = 1$

\# Re-write the constraint

$g(x,y) = x^2+y^2-1 = 0$

\# Set up Lagrange func

$\mathcal{L}(x,y,\lambda) = xy - \lambda(x^2+y^2-1)$

\# Find the crit pts

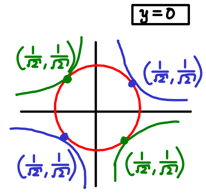
$\begin{cases} \mathcal{L}_x = y - \lambda(2x) = 0 \\ \mathcal{L}_y = x - \lambda(2y) = 0 \\ \mathcal{L}_\lambda = -(x^2+y^2-1) = 0 \end{cases} \iff \begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2+y^2 = 1 \end{cases}$

$x = 2\lambda(2\lambda x) = 4\lambda^2 \cdot x$

Either $\boxed{x=0}$ ⇓ $\boxed{y=0}$ OR $\boxed{1 = 4\lambda^2}$ ⇓ $\boxed{\lambda = \pm\frac{1}{2}}$ ⇓ $\boxed{y=-x}$ $\boxed{y=x}$

$(x,y) = (0,0)$ is not on the circle, contradiction

We get:	
$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$
$(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$



Ex: (§3.4 Q30)

Suppose $S \subseteq \mathbb{R}^3$ is a surface defined by $f(x,y,z) = 1$
Suppose P is a pt on S s.t the distance to the origin is maximal
Claim: $\vec{OP}$ is perpendicular to S

Consider maximizing $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ on the surface $f(x,y,z) = 1$

The Lagrange function is
$\mathcal{L}(x,y,z,\lambda) = d - \lambda(f-1)$

If we are a crit pt:
$\mathcal{L}_x = d_x - \lambda f_x = 0$
$\mathcal{L}_y = d_y - \lambda f_y = 0$
$\mathcal{L}_z = d_z - \lambda f_z = 0$

diff. d
$\begin{cases} kx = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \lambda f_x \\ ky = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \lambda f_y \\ kz = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \lambda f_y \end{cases}$

We obtain: $k\vec{OP} = \nabla f$ and we know ∇f is perpendicular to S

Summary
Constrained Optimization
"Check the endpoints (boundary)"
Bounded Sets
Closed Sets
Extreme Value Theorem
Lagrange Multipliers
"Find max/min using level curves"