Integration (§5.2)

Recall,

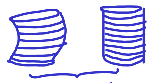
$\int_a^b f(x)dx =$ "signed area of $f(x)$ on $[a,b]$"

Fact (Fundamental Theorem of Calculus)
$\int_a^b f(x)dx = F(b) - F(a)$ where
$F'(x) = f(x)$
F is an anti-derivative of f

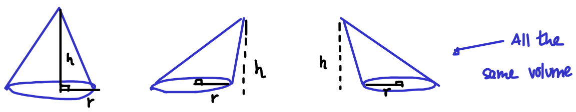
Cavalleri's Principle

"The integral of area is volume."

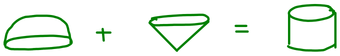
Corollary: If two bodies have the same horizontal slice area functions then they have the same volume.



Ex: All three cones have the same volume



Application of Cavalleri

Claim: 

The volume of a hemi-sphere of radius r and a cone of radius r and height r equals the volume of a cylinder of height and radius r.

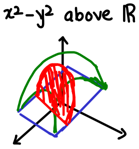
The hemi-sphere is $x^2+y^2+z^2 = r^2$

cone is $z = \sqrt{x^2+y^2}$

Ex: Find the volume under $z = 2-x^2-y^2$ above $\mathbb{R} = \{(x,y) \ -1 \le x \le 1 \text{ and } -1 \le y \le 1\}$
$= [-1,1] \times [-1,1]$

Setup an Integral

$V = \int_{-1}^{1} \text{Area}(x)dx$



$= \int_{-1}^{1} \boxed{\int_{-1}^{1} 2-x^2-y^2 dy} dx$

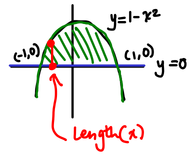
$= \int_{-1}^{1} \left[2y - x^2y - \frac{1}{3}y^3\right]_{-1}^{1} dx$

$= \int_{-1}^{1} \left[\left(2\cdot 1 - x^2(1) - \frac{1}{3}(1)^3\right) - \left(-2 + x^2 + \frac{1}{3}\right)\right] dx$

$= \int_{-1}^{1} 4 - 2x^2 - \frac{2}{3} dx = \int_{-1}^{1} \frac{10}{3} - 2x^2 dx$

$= \left[\frac{10}{3}x - \frac{2}{3}x^3\right]_{-1}^{1} = \left(\frac{10}{3} - \frac{2}{3}\right) - \left(\frac{-10}{3} + \frac{2}{3}\right) = \frac{16}{3}$

Ex: Find the area bounded by $y = 1-x^2$ and $y=0$



$A = \int_{-1}^{1} \text{Length}(x)dx$

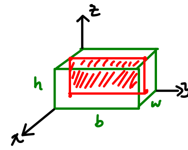
$= \int_{-1}^{1} 1-x^2 dx$

$= \left[x - \frac{1}{3}x^3\right]_{-1}^{1}$ # Using FTC

$= \left(1 - \frac{1}{3}(1)^3\right) - \left(1 - \frac{1}{3}(-1)^3\right)$

$= \frac{2}{3} - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$

Ex: Find the volume of a rectangular prism



Let h = height, b = base, w = width

Setup an integral

$\text{Vol} = \int_0^w \text{Area}(x)dx$

$= \int_0^w bh\,dx = [bhx]_0^w$

$= bhw - bh\cdot 0 = bhw$

Find the area of cross-section

At $z=k$ we get: $x^2+y^2+k^2 = r^2 \Leftrightarrow x^2+y^2 = r^2-k^2$ ← For hemi-sphere

A circle of radius $\sqrt{r^2-k^2}$

At $z=k$ we get: $k = \sqrt{x^2+y^2} \Leftrightarrow k^2 = x^2+y^2$ ← For cone

A circle of radius k

Adding together areas

$\pi k^2 + \pi\left(\sqrt{r^2-k^2}\right)^2 = \pi k^2 + \pi(r^2-k^2) = \pi r^2$

> To find the volume under $z = f(x,y)$ on the region $[a,b]\times[c,d]$
>
> $V = \int_a^b \text{Area}(x)dx = \int_a^b \boxed{\int_c^d f(x,y)dy} dx$

Double Integrals (§ 5.3)

Ex: Calculate $\int_0^1\int_2^4 x+y\,dx\,dy$

This is the volume under $z=x+y$ above the rectangle $[2,4]\times[0,1]$

$\int_0^1\int_2^4 x+y\,dx\,dy$
$=\int_0^1\left[\frac{1}{2}x^2+yx\right]_2^4 dy$
$=\int_0^1\left[\frac{1}{2}4^2+4y\right]-\left[\frac{1}{2}2^2-2y\right]dy$
$=\int_0^1 8+4y-2-2y\,dy$
$=\int_0^1 6+2y\,dy$
$=\left[6y+y^2\right]_0^1=6+1-0=7$ //

"What about $dy\,dx$?"

#Switch the bounds of integration

$\int_2^4\int_0^1 x+y\,dy\,dx$
$=\int_2^4\left[xy+\frac{1}{2}y^2\right]_0^1 dx$
$=\int_2^4 x+\frac{1}{2}dx$
$=\left[\frac{1}{2}x^2+\frac{1}{2}x\right]_2^4$
$=\left(\frac{1}{2}\cdot4^2+\frac{1}{2}\cdot4\right)-\left(\frac{1}{2}\cdot2^2+\frac{1}{2}\cdot2\right)$
$=(8+2)-(2+1)=7$ //

Observe: $dy\,dx$ and $dx\,dy$ give the same volume

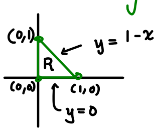
Integration on y-SIMPLE regions

$\iint_R f(x,y)\,dA$ ← dA means "wrt Area" $dA=dx\,dy=dy\,dx$

$=\int_a^b\left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx$

Ex: Integrate $f(x,y)=xy$ over the triangle w/ verticles $(0,0)$, $(0,1)$, $(1,0)$

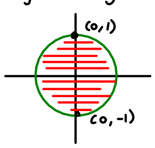
Sketch the region



$\iint_R f(x,y)\,dA=\int_0^1\int_0^{1-x} xy\,dy\,dx=\int_0^1\left[\frac{1}{2}xy^2\right]_0^{1-x}dx$
$=\int_0^1\frac{1}{2}x(1-x)^2dx$
$=\int_0^1(x-2x^2+x^3)dx$
$=\left[\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^1$
$=\frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{1}{2}\left(\frac{6}{12}-\frac{8}{12}+\frac{3}{12}\right)=\frac{1}{24}$ //

Ex: Setup the integral of $f(x,y)$ on $R=\{(x,y): x^2+y^2\le1\}$ using $dA=dx\,dy$

#Draw region



$\iint_R f(x,y)\,dx\,dy=\int_{-1}^1\left[\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\right]dy$

#solve for x in terms of y

$x^2+y^2=1 \Rightarrow x^2=1-y^2 \Rightarrow x=\pm\sqrt{1-y^2}$

Volume:

Ex: Find the volume bounded by $x^2+z^2=r^2$ and $y^2+z^2=r^2$ (circles)

By Cavalleri's Principle

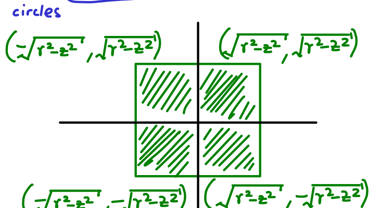
Volume $=\int_{-r}^r \text{Area}(z)\,dz$

#Find Area(z)

$x^2=r^2-z^2 \Rightarrow x=\pm\sqrt{r^2-z^2}$

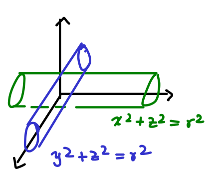
$y^2=r^2-z^2 \Rightarrow y=\pm\sqrt{r^2-z^2}$

Region in the plane at the slice z



$\text{Area}(z)=\left[\sqrt{r^2-z^2}-\left(-\sqrt{r^2-z^2}\right)\right]\left[\sqrt{r^2-z^2}-\left(-\sqrt{r^2-z^2}\right)\right]$
$=2\left(\sqrt{r^2-z^2}\right)\cdot2\left(\sqrt{r^2-z^2}\right)$

Volume $=\int_{-r}^r 4(r^2-z^2)\,dz$
$=\left[4\left(r^2z-\frac{z^3}{3}\right)\right]_{-r}^r$
$=4\left(r^3-\frac{r^3}{3}\right)-4\left(-r^3+\frac{r^3}{3}\right)$
$=\frac{16}{3}r^3$



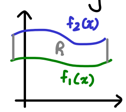
Summary

- Double Integrals
- Cavalleri's Principle (Vol = ∫Area)
- Fubini's Thm ($dA=dx\,dy=dy\,dx$)
- x-SIMPLE, y-SIMPLE, SIMPLE
- Achimedes ◒ + ◬ = ⌭

Now we integrate over more general regions in the plane

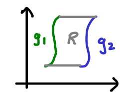
Defⁿ: A region R is y-SIMPLE ("y co-ordinate is simple") if there are cont. fxns $f_1(x)$ and $f_2(x)$

$R\{(x,y): x\in[a,b]\Rightarrow f_1(x)\le y\le f_2(x)\}$

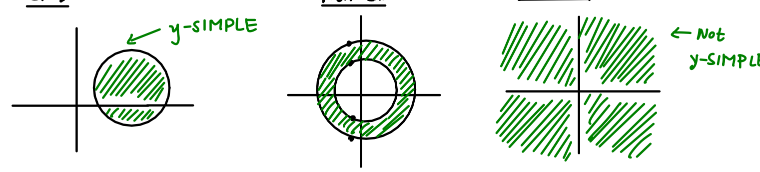


Defⁿ: A region is x-SIMPLE

$R\{(x,y): y\in[a,b]\Rightarrow g_1(y)\le x\le g_2(y)\}$ (continuous)



Ex: Non-Ex: Non-Ex:

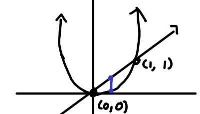


Defⁿ: R is SIMPLE if it x-SIMPLE and y-SIMPLE.

? Bounded by cont. fxns in both directions

Ex: Setup the integral of $f(x,y)$ over the region bounded by $y=x^2$ & $y=x$ using both $dA=dy\,dx$ and $dA=dx\,dy$

Sketch the region



$dA=dy\,dx$

$\iint_R f(x,y)\,dy\,dx=\int_0^1\left[\int_{x^2}^{x} f(x,y)\,dy\right]dx$

Determine bounds of int

$x=x^2 \Leftrightarrow x^2-x=0 \Leftrightarrow x(x-1)=0$

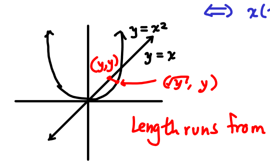
$dA=dx\,dy$

$\iint_R f(x,y)\,dx\,dy=\int_0^1\left[\int_y^{\sqrt{y}} f(x,y)\,dx\right]dy$

"how far does x go at height y?"

Solve for x in terms of y

$y=x^2 \Rightarrow \pm\sqrt{y}=x \Rightarrow x=\sqrt{y}$

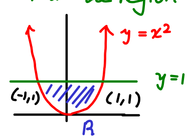


Area as Integral

We can calculate area as the volume of a prism of height one

Ex: Find the area bounded by $y=1$ and $y=x^2$

#Draw the region



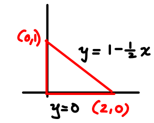
Area $=\iint_R 1\,dA$

$=\int_{-1}^1\int_{x^2}^1 1\,dy\,dx=\int_{-1}^1[y]_{x^2}^1dx=\int_{-1}^1 1-x^2dx=4/3$ //

Fubini's Theorem

Fact: If f is C^2 and R is simple $\iint_R f(x,y)\,dx\,dy=\iint_R f(x,y)\,dy\,dx=\iint_R f(x,y)\,dA$

Ex: Calculate the area under $z=2-x-y$ and above the triangle w/ vertices $(0,0)$, $(2,0)$, $(0,1)$ using $dA=dx\,dy=dy\,dx$



#$dA=dy\,dx$

Vol $=\iint_R 2-x-y\,dA=\int_0^2\int_0^{1-\frac{1}{2}x}2-x-y\,dy\,dx=\int_0^2\left[2y-xy-\frac{y^2}{2}\right]_0^{1-\frac{1}{2}x}dx$

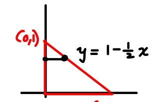
$=\int_0^2\left[2(1-\frac{1}{2}x)-x(1-\frac{1}{2}x)-\frac{1}{2}(1-\frac{1}{2}x)^2\right]dx=\int_0^2 2-x-x-\frac{1}{2}x^2-\frac{1}{2}(1-x+\frac{1}{4}x^2)dx$

$=\int_0^2\frac{3}{2}-\frac{3}{2}x+\frac{3}{8}x^2dx=\left[\frac{3}{2}x-\frac{3}{4}x^2+\frac{1}{8}x^3\right]_0^2=3-3+1=1$

By Fubini, $1=\iint_R 2-x-y\,dx\,dy$

#$dA=dx\,dy$

Vol $=\int_0^1\left[\int_0^{2-2y}2-x-y\,dx\right]dy$



$y=1-\frac{1}{2}x \Leftrightarrow 2y=2-x$

$x=2-2y$