Basic Training in Euclidean Geometry

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1 Advice and Suggestions

- Draw big diagrams. Lots of them.
- Use multiple colours to keep track on information.
- Draw several examples with different lengths/angles.
- Avoid coordinates.

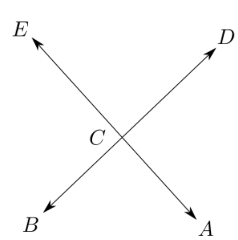
2 Facts and Questions

Fact 1. *Congruence conditions for triangles: (SAS, ASA, SSS, AAS) Side-angle-side, angle-side-angle, side-side-side, angle-angle-side.*

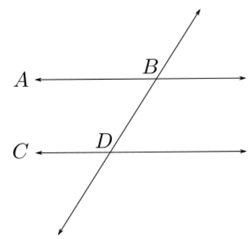
Question 1. *Why is angle-side-side not a congruence condition?*

Fact 2. *If an angle $\angle A$ measures a straight line, then $\angle A = \pi = 180°$.*

Question 2. *Show that $\angle ECD = \angle ACB$ using the Fact 2.*



Fact 3 (Transversals[1]). *AB and CD are parallel iff* $\angle B = \angle D$.



Question 3. *Show that the sum of the interior angles of a triangle is* π. *(Idea: Add a new parallel line.)*

Question 4. *For any triangle* $\triangle ABC$*, the exterior angle at* A *is the sum of the interior angles at* B *and* C.

Question 5 (Parallelograms). *Use congruence and parallels to show:*

- *The opposite sides of a parallelogram have the same length.*
- *The opposite angles of a parallelogram are equal, and adjacent angles are "supplementary" (sum to* π*).*
- *The diagonals of a parallelogram bisect each other.*
  (Idea: Introduce triangles.)

Question 6 (Isoceles triangles). *A triangle* $\triangle ABC$ *is isoceles if* $|AB| = |AC|$. *Show that: If* $\triangle ABC$ *is isoceles then* $\angle B = \angle C = 90° - (\angle A)/2$.

Question 7. *Let* $\triangle ABC$ *be a triangle. Let* D*,* E*, and* F *be the midpoints of* BC*,* AC*, and* AB *respectively. Show that the lines:* DE*,* EF*, and* FD*, dissect* $\triangle ABC$ *in to congruent triangles.*

(Big Idea: Introduce parallels to BC *and* AC *at* D*. These will create new "phantom points" on* BC *and* AC*. These new points will be very helpful because of our theory of parallelograms.)*

[1] Fact 3 is a really remarkable property of Euclidean geometery. It does not hold in spherical or hyperbolic geometry where weirder things are true.

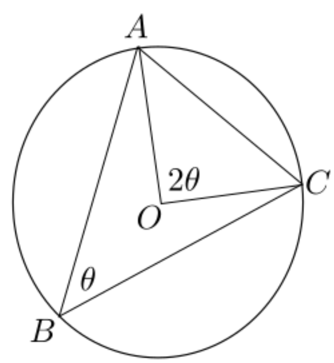
Fact 4 (Circles and Chords)**.** *If any two are true, then all three are true.*

- *The line passes through the center of the circle.*
- *The line passes through the midpoint of the chord.*
- *The line line is perpendicular to the chord.*

Fact 5 (Circles and Tangents)**.** *A tangent to a circle is perpendicular to the radius at the point of tangency. Also, a perpendicular to a tangent line placed at the point of tangency, will pass through the center of the center of the circle.*

Question 8. *Suppose that A is outside a circle. If AX and AY are tangent to the circle, then $|AX| = |AY|$.*

Question 9 (Inscribed Angle Theorem)**.** *Consider a circle centered at O with points A, B, C on the perimeter of the circle. Show that $\angle ABC = 2\angle AOC$. (Idea: Chase angles.)*



Question 10. *Consider a semi-circle with base AC.*
If B is on the perimeter of the semi-circle, show that $\angle ABC = 90°$.

Question 11 (Circumcircles)**.** *The circumcircle of a triangle $\triangle ABC$ is a circle passing through A, B, and C.*

- *Suppose that a circumcircle exists. Show that its center is the intersection point of the perpendicular bisectors of the sides of the triangle. (If a circumcircle exists, then it has a unique center.)*

- *Given a triangle, any two perpendicular bisectors will intersect in a point equidistant from all three vertices. (Any triangle has a circumcenter.)*

- *All three perpendicular bisectors intersect in a unique point, the "circumcenter".*

Question 12 (Incircles). *The incircle of a triangle $\triangle ABC$ is a circle tangent to AB, BC, and AC.*

- *Suppose that an incircle exists. Show that its center is the intersection of any two angle bisectors.*

- *Show that the intersection of any two angle bisectors is the center of an incircle and this point is unique.*

- *All three angle bisectors intersect in a unique point, the "incenter".*

3 Contest Problems

Question 13 (Canada 1991). *Let C be a circle and P a point in the plain. Each line through P that interesects C determines a chord of C. Show that the midpoints of these chords form a circle.*

Question 14 (Rochester 2012). *Let $\triangle ABC$ be an isoceles triangle with $AC = BC$ and $\angle ACB = 80°$. Consider and interior point of this triangle such that $\angle MBA = 30°$ and $\angle MAB = 10°$. Find with proof, the measure $\angle AMC$.*

Question 15 (Rochester 2013). *Let $\triangle ABC$ be a triangle with $\angle BAC = 120°$. Suppose the bisectors of $\angle BAC$, $\angle ABC$, and $\angle ACB$ meet BC, AC, and AB at points D, E, F, respectively. Prove that $\triangle DEF$ is a right angled triangle.*