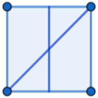
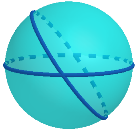
MAT 402: Classical Geometry

Groups

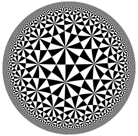


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

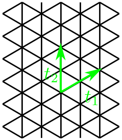
Spherical



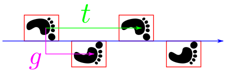
Hyperbolic



Tilings



Friezes

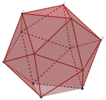


Trigonometry

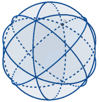
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

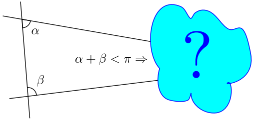
Platonic Solids



Coxeter



Parallels



MAT 402: Monday November 23rd 2020

Learning Objectives:

- Show the invariance of cross-ratios under $\mathbb{R}\text{Möb}$
- Use the Möbius distance to calculate lengths in the complex plane.

Cross-Ratios

Definition

The cross-ratio of four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ is:

$$\langle z_1, z_2, z_3, z_4 \rangle = \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2} = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$$

Theorem

Cross-ratios are invariant under $f \in \mathbb{R}M\ddot{o}b$.

$$f \in \mathbb{R}M\ddot{o}b \Rightarrow \langle f(z_1), f(z_2), f(z_3), f(z_4) \rangle = \langle z_1, z_2, z_3, z_4 \rangle$$

Fractional Linear Transformations

Lemma

$$f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d-d)+b}{cz+d} = \frac{a}{c}\cdot 1 + \frac{-ad+bc}{c(cz+d)}$$

This is a composition of maps of the form: $f_\times(z) = Kz$, $f_+ = z+K$, *and* $f_1(z) = 1/z$.

Task

Show that cross-ratios are invariant under maps of the form $f_\times(z)$.

$$\begin{aligned}
\langle f_\times(z_1), f_\times(z_2), f_\times(z_3), f_\times(z_4)\rangle &= \frac{Kz_3 - Kz_1}{Kz_3 - Kz_2} : \frac{Kz_4 - Kz_1}{Kz_4 - Kz_2} \\
&= \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2} \\
&= \langle z_1, z_2, z_3, z_4\rangle
\end{aligned}$$

Fractional Linear Transformations

Task

Show that cross-ratios are invariant under maps of the form $f_+(z)$.

$$
\begin{aligned}
\langle f_+(z_1), f_+(z_2), f_+(z_3), f_+(z_4)\rangle &= \frac{(z_3+K)-(z_1+K)}{(z_3+K)-(z_2+K)} : \frac{(z_4+K)-(z_1+K)}{(z_4+K)-(z_2+K)} \\
&= \frac{z_3-z_1}{z_3-z_2} : \frac{z_4-z_1}{z_4-z_2} \\
&= \langle z_1, z_2, z_3, z_4\rangle
\end{aligned}
$$

Fractional Linear Transformations

Task

Show that cross-ratios are invariant under maps of the form $f_1(z)$.

$$
\begin{aligned}
\langle f_1(z_1), f_1(z_2), f_1(z_3), f_1(z_4)\rangle &= \frac{1/z_3 - 1/z_1}{1/z_3 - 1/z_2} : \frac{1/z_4 - 1/z_1}{1/z_4 - 1/z_2} \\
&= \frac{1/z_3 - 1/z_1}{1/z_3 - 1/z_2}\left(\frac{z_1 z_2 z_3}{z_1 z_2 z_3}\right) : \frac{1/z_4 - 1/z_1}{1/z_4 - 1/z_2}\left(\frac{z_1 z_2 z_4}{z_1 z_2 z_4}\right) \\
&= \frac{z_1 z_2 - z_2 z_3}{z_1 z_2 - z_1 z_3} : \frac{z_1 z_2 - z_2 z_4}{z_1 z_2 - z_1 z_4} \\
&= \frac{z_2(z_1 - z_3)}{z_1(z_2 - z_3)} : \frac{z_2(z_1 - z_4)}{z_1(z_2 - z_4)} = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4} \\
&= \langle z_1, z_2, z_3, z_4\rangle
\end{aligned}
$$

The Möbius Distance

Theorem

Cross-ratios are invariant under $f \in \mathbb{R}M\ddot{o}b$.

$$f \in \mathbb{R}M\ddot{o}b \Rightarrow \langle f(z_1), f(z_2), f(z_3), f(z_4)\rangle = \langle z_1, z_2, z_3, z_4\rangle$$

Definition

The Möbius distance between two points $X, Y \in \mathbb{C}_+$ is:

$$\mu(A, B) = |\ln(|\langle A, B, X, Y\rangle|)|$$

where X and Y are the intersection points of the geodesic joining AB with the absolute $\mathbb{A}$.

The Möbius Distance

Task

Find the hyperbolic distance between $A = i$ and $B = 2i$.

$$\begin{aligned}
\mu(A,B) &= |\ln(|\langle A,B,X,Y\rangle|)| = |\ln(|\langle i,2i,\infty,0\rangle|)| \\
&= \left|\ln\left(\left|\frac{i-\infty}{2i-\infty} : \frac{i-0}{2i-0}\right|\right)\right| \\
&= \left|\ln\left(\left|1 : \frac{1}{2}\right|\right)\right| = \ln(2) \simeq 0.693
\end{aligned}$$

Lemma

In general we have: $\mu(i, Ki) = |\ln(K)|$. Thus, distances are exponentially distorted.

The Möbius Distance

Task

Find the hyperbolic distance between $A = e^{i\pi/3} = \omega$ and $B = e^{i2\pi/3} = \omega^2$.

We have:

$$\begin{aligned}
\mu(A,B) &= |\ln(|\langle A,B,X,Y\rangle|)| = |\ln(|\langle \omega,\omega^2,1,-1\rangle|)| \\
&= \left|\ln\left(\left|\frac{\omega-1}{\omega^2-1} : \frac{\omega+1}{\omega^2+1}\right|\right)\right| = \left|\ln\left(\left|\frac{(\omega-1)(\omega^2+1)}{(\omega^2-1)(\omega+1)}\right|\right)\right| \\
&= \left|\ln\left(\left|\frac{\omega^3+\omega-\omega^2-1}{\omega^3+\omega^2-\omega-1}\right|\right)\right| = \left|\ln\left(\left|\frac{-1+0}{-1-2}\right|\right)\right| \\
&= |\ln(1/3)| = |-\ln(3)| = \ln(3)
\end{aligned}$$