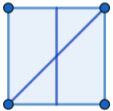
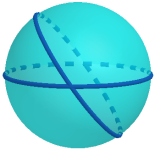
MAT 402: Classical Geometry

Groups

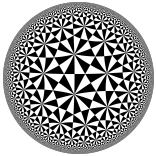


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

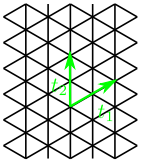
Spherical



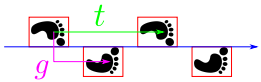
Hyperbolic



Tilings



Friezes

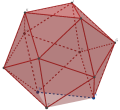


Trigonometry

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

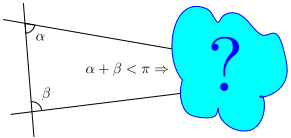
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Platonic Solids



Coxeter

Parallels

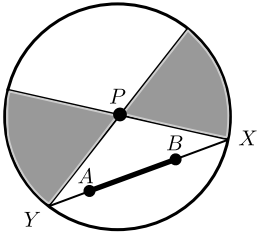


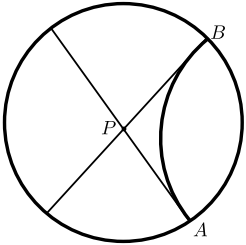
MAT 402: Friday November 29th 2020

Learning Objectives:

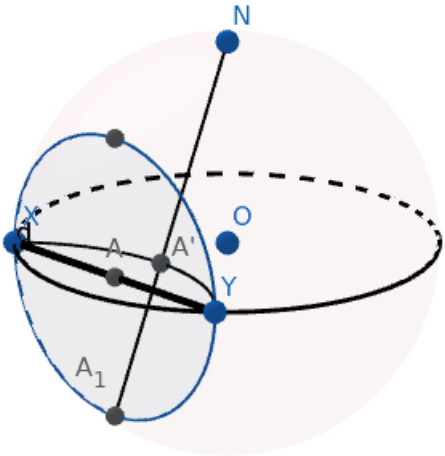
- Isomorphism of the Cayley-Klein and Poincaré disk models
- Isomorphism of the two Poincaré models
- Hyperbolic trigonometry functions
- Absolute constants

Cayley and Poincaré





Cayley and Poincaré



https://www.geogebra.org/3d/gzzatbga

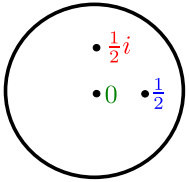
The chord XY is a geodesic in the Cayley-Klein model.

The arc $XA'Y$ is the corresponding geodesic in the Poincaré model.

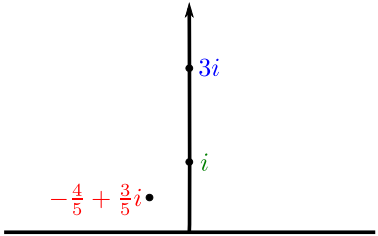
The Poincaré Disk and Half-Plane

Consider the map $\Omega : \mathbb{D}^2 \to \mathbb{C}_+$.

$$\Omega(z) = i \cdot \frac{1+z}{1-z}$$



$\mathbb{D}^2$



$\mathbb{C}_+$

It induces an isomorphism of geometries:

$$(\mathbb{H}^2, \mathcal{M}) \simeq (\mathbb{C}_+, \mathbb{R}\text{Möb})$$

The Hyperbolic Model Isomorphism Theorem

Task

Show that $\Omega : \mathbb{D}^2 \to \mathbb{C}_+$ *given by* $\Omega(z) = i\frac{1+z}{1-z}$ *satisfies:*

$$|z| < 1 \iff \Re\left(i\frac{1+z}{1-z}\right) > 0$$

The Hyperbolic Model Isomorphism Theorem

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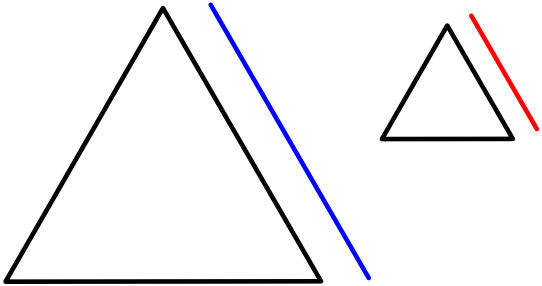
Corollary (10.2.3)

The three models of hyperbolic geometry, namely the Poincaré disk and half-plane models and the Cayley-Klein model, are isomorphic as geometries.

Absolute Constants

In Euclidean space, there is no "natural" unit of measurement.
Similarity of figures implies there is no natural reference point.



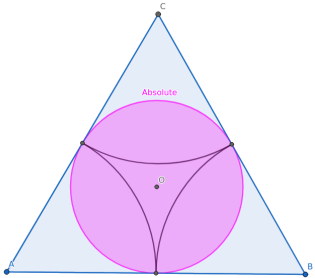
In hyperbolic geometry, there are natural units!

The Ideal Triangle

We know that the area of a hyperbolic triangle is:

$$S = \pi - \alpha - \beta - \gamma \Rightarrow S \leq \pi$$

Thus, the largest triangle in hyperbolic space has area $S = \pi$.

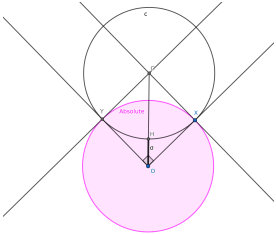


The natural unit of measure for area is π.

The Schweikart Radius

Definition

Given a pair of perpendicular rays OA and OB they meet the absolute $\mathbb{A}$ at points X and Y. There is a unique line XY passing through X and Y. The distance from O to XY is the *Schweikart radius* denoted σ.



The natural unit of measure for length is σ.

Task

What happens if we try to do this in Euclidean space or hyperbolic space?

Computing the Scheikart Radius

Task

Compute the Scheikart radius using the Möbius distance.