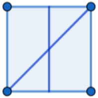
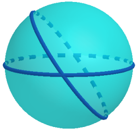
MAT 402: Classical Geometry

Groups

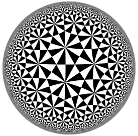


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

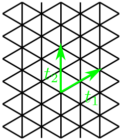
Spherical



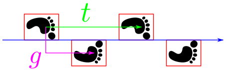
Hyperbolic



Tilings



Friezes

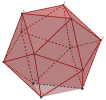


Trigonometry

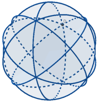
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

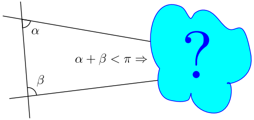
Platonic Solids



Coxeter



Parallels



Any question about the homework? Comments?

What is something neat you learned in another class?

News flash: Slides are online. Piazza is up!

MAT 402: Friday September 25st 2020

Learning Objectives:

- Provide examples of finite subgroups of SO(3)
- Explain the geometric significance of orbits and stabilizers in $\mathbb{R}^3$.
- Distinguish between full and rotational symmetry groups

The Special Orthogonal Group

Definition

The special orthogonal group SO(3) is the group of linear maps $T : \mathbb{R}^3 \to \mathbb{R}^3$ which preserve orthogonal bases and orientation. Alternatively, it is the group of rotations of three dimensional space.

Task (5 min)

Is SO(3) commutative? Does $ST = TS$ for all $S, T \in \mathrm{SO}(3)$?

The Special Orthogonal Group

Task (2 min)

If $T \in \mathrm{SO}(3) \setminus \{I_{3\times 3}\}$ *acts on* $\mathbb{R}^3$ *then how many fixed points does* T *have?*
Suppose, T *acts on* $\mathbb{S}^2$*. How many fixed points does this action have?*

The Special Orthogonal Group

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Definition

We call the two fixed points of T acting on $\mathbb{S}^2$ the poles of T.

The Special Orthogonal Group

Task (2 min)

Consider $T = \begin{pmatrix} \cos(\pi/4) & -\sin(pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Let* $G = \langle T \rangle \subseteq \mathrm{SO}(3)$.

What is G *isomorphic to? What is* $\mathrm{Orb}(1,0,0)^T$*? What is* $\mathrm{St}(0,0,\pm 1)^T$?

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Question

What are some other finite subgroups of $\mathrm{SO}(3)$*?*

Fixed Point Counting

Lemma

If $G^+ \subseteq SO(3)$ *then* G^+ *maps poles to poles.*

Fixed Point Counting

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Proof.

Suppose that $p \in \mathbb{S}^2$ is a pole of T. Let $S \in G^+$ be arbitrary.
pS is a fixed point of $S^{-1}TS$ because:

$$pS(S^{-1}TS)$$

Fixed Point Counting

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Thus, pS is a pole. □

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Theorem

If p is the pole of an element $T \in G^+ \subset \mathrm{SO}(3)$ of maximal order k and $|G^+| = n$ then $\mathrm{Orb}(p) = n/k$.

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$$G^+/\,\mathrm{St}(p) = \{\mathrm{St}(p)g_1, \mathrm{St}(p)g_2, \ldots, \mathrm{St}(p)g_q\}$$

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Task (5 min)

Discuss this proof in small groups. (What's missing?)

Square Bipyramid

Task (5 min)

Consider $S = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 2)\}$. *What is the rotation group of* S*? What are the poles of* S*? What are the orbits of the poles of* S*?*