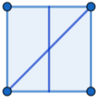
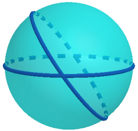
MAT 402: Classical Geometry

Groups

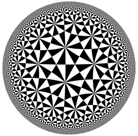


$$\mathrm{Symm}(\square) = \langle r, s : r^2 = s^2 = (rs)^4 = e \rangle$$

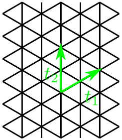
Spherical



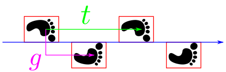
Hyperbolic



Tilings



Friezes

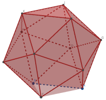


Trigonometry

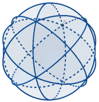
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

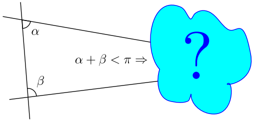
Platonic Solids

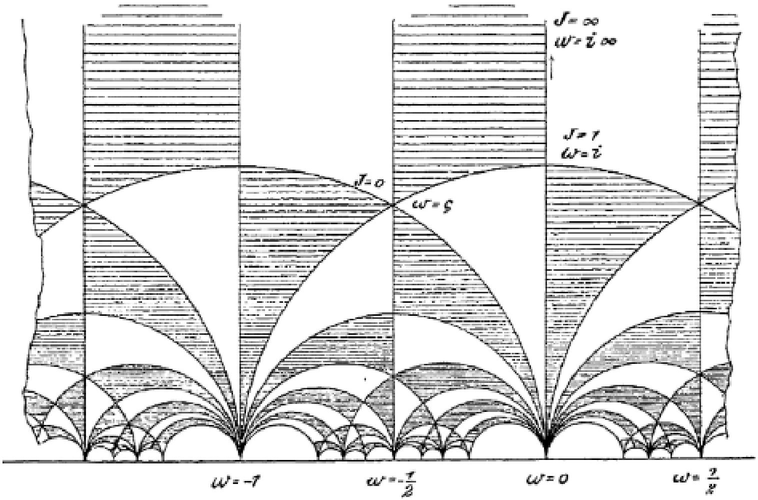


Coxeter



Parallels





From Klein and Fricke - Theorie der Elliptischen Modulfunction (1927)

MAT 402: Friday November 13th 2020

Learning Objectives:

- Use complex coordinated to describe the upper-half plane model.
- Find fractional-linear transformations with specific properties.

The Complex Plane

Definition

The complex numbers are $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ and $i^2 = -1$.
We can write $a + bi = Re^{i\theta}$ with $R > 0$ and $0 \leq \theta < 2\pi$ using $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
The real part of $a + bi$ is $\Re(a + bi) = a$ and the imaginary part is $\Im(a + bi) = b$.

Task

Describe the transformation $f(z) = zRe^{i\theta}$ geometrically.

The Riemann Sphere

Definition

The *Riemann sphere* or *extended complex plane* is: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Definition

A *fractional linear transformation* is a map:

$$f(z) = \frac{az + b}{cz + d}$$

where $cb - ad \neq 0$ which is defined on $\mathbb{C} \setminus \{-d/c\}$.
We extend it to $\overline{\mathbb{C}}$ by defining: $f(-d/c) = \infty$ and $f(\infty) = a/c$.

Task (Don't show this slide to the first years!)

What is $f(\infty)$ *if* $f(z) = \dfrac{2z + 1}{3z - 1}$*?*

The Group $\mathbb{R}\mathrm{M\ddot{o}b}$

Definition

The *upper half-plane* is $\mathbb{C}_+ = \{x + iy : y > 0\}$. The *absolute* is $\mathbb{A} = \{x + i0\}$.

Theorem

A fractional linear map of the form:

$$\frac{az+b}{cz+d} \quad or \quad \frac{a\bar{z}+b}{c\bar{z}+d}$$

where $a, b, c, d \in \mathbb{R}$ and $bc - ad > 0$ preserves the upper-half plane.
All such transformations form the group $\mathbb{R}M\ddot{o}b$ of real Möbius transformations.

Definition

The Poincaré Half-Plane model is $(\mathbb{C}_+ : \mathbb{R}\mathrm{M\ddot{o}b})$.

$\mathbb{R}$Möb Preserves the Half-Plane

Task

Show that $\mathbb{R}M\ddot{o}b$ preserves the upper-half plane.

If $\Im(z) > 0$ *and* $bc - ad > 0$ *then* $\Im\left(\frac{az+b}{cz+d}\right) > 0$

$\mathbb{R}$Möb is a Group

Task

Show that $\mathbb{R}M\ddot{o}b$ is closed under composition: $f(z), g(z) \in \mathbb{R}M\ddot{o}b \Rightarrow f(g(z)) \in \mathbb{R}M\ddot{o}b$

Task

Show that $\mathbb{R}M\ddot{o}b$ is closed under inversion: $f(z) \in \mathbb{R}M\ddot{o}b \Rightarrow f^{-1}(z) \in \mathbb{R}M\ddot{o}b$

The Action of the $\mathbb{R}\mathrm{M\ddot{o}b}$

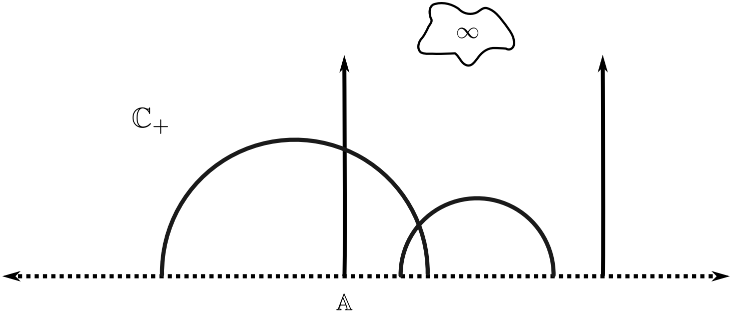
Task

Find a fractional linear transformation mapping $\varphi(0) = 1$, $\varphi(1) = \frac{1}{2}$, $\varphi(\infty) = 1$.

The Poincaré Half-Plane Model

Definition

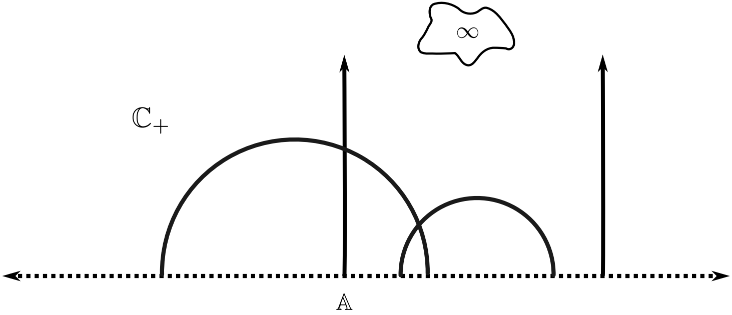
Geodesics in $(\mathbb{C}_+ : \mathbb{R}\mathrm{M\ddot{o}b})$ are vertical lines or half-circles perpendicular to $\mathbb{A}$.



Task

Find an isometry $f \in \mathbb{R}M\ddot{o}b$ which fixes the geodesic $\gamma_1(t) = ti$ for $t \in \mathbb{R}$.

The Poincaré Half-Plane Model



Task

Find an isometry $f \in \mathbb{R}M\ddot{o}b$ *which fixes the geodesic:* $\gamma_2(t) = \cos(t) + i\sin(t)$ *for* $t \in (0, \pi)$.