Week 2: Review of Linear Algebra

The Standard Basis of $\mathbb{R}^3$ (M&T p. 9)

In your handwritten notes, you should write $\vec{x}$ for $\mathbf{x}$.

$\vec{x}$

We have two notations for the standard basis of $\mathbb{R}^3$.

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Context: The $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ notation is preferred by physicists and engineers.

Example: Vector Addition and Scaling

- Add the vectors $\mathbf{v}_1 = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v}_2 = [1, 2, 3]^T$. $= i + 2j + 3k$
- Scale the vector $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$ by two and write the result using the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

$$\begin{aligned} v_1 + v_2 &= (i + j - 2k) + (i + 2j + 3k) \\ &= (1+1)i + (1+2)j + (-2+3)k \\ &= 2i + 3j + 1k \end{aligned}$$

$$\begin{aligned} 2v_3 &= 2(2v_1 + 3v_2) = 4v_1 + 6v_2 \\ &= 4(i + j - 2k) + 6(i + 2j + 3k) \\ &= (4+6)i + (4+12)j + (-8+18)k \\ &= 10i + 16j + 10k \end{aligned}$$

i vs $\vec{i}$ are both fine.

Wikipedia: Quaternions

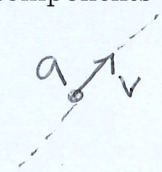
Point-Direction Form of Lines (M&T p. 12)

Suppose that $\mathbf{v}$ is a non-zero vector. A line ℓ through point $\mathbf{a}$ pointing in direction $\mathbf{v}$ can be written

$$\ell(t) = \mathbf{a} + t\mathbf{v}$$

In three-dimensional space we can write $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{v} = (a, b, c)$ in components and obtain:

$$\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}$$



Example: Find a Line

Find the equation of the line through the points $(1, 1, 2)$ and $(1, 2, 3)$.
Write your answer in point-direction format.

We pick $a = (1,1,2)$ ← we picked a value of a

and $v = (1,2,3) - (1,1,2) = (0,1,1)$

This gives $\ell(t) = a + tv = (1,1,2) + t(0,1,1)$

We could pick $a = (1,2,3)$

and $v = (1,1,2) - (1,2,3) = (0,-1,-1)$

This gives

$$\ell(t) = (1,2,3) + t(0,-1,-1)$$

BOTH OF THESE DEFINE THE SAME LINE

Example: Do These Lines Intersect?

Consider the following lines $\ell_1(t) = (1,2,3) + t(1,0,0)$ and $\ell_2(s) = (-5,-2,1) + s(1,0,2)$ in $\mathbb{R}^3$. Determine whether the lines ℓ_1 and ℓ_2 intersect.

Lines are given by linear systems.

$$\ell_1 \begin{cases} x = 1 + 1\cdot t = 1+t \\ \boxed{y = 2 + 0\cdot t = 2} \\ z = 3 + 0\cdot t = 3 \end{cases}$$

$$\ell_2 \begin{cases} x = -5 + 1\cdot s = -5+s \\ \boxed{y = -2 + 0\cdot s = -2} \\ z = 1 + 2\cdot s = 1+2s \end{cases}$$

Notice: Both y values are constant and unequal therefore the lines will NOT intersect.

For context: If all equations of ℓ_1 involve t and all equations of ℓ_2 involve s then set corresponding components equal and solve the linear system.

Parametric Equations of Lines (M&T p. 15)

A parametric equation for the line ℓ through $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is:

$$\begin{cases} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{cases}$$

Notice: This is the Point-Direction Form with $\mathbf{v} = Q - P = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Example: Parametric and Point-Direction

Consider the line passing through $(1, 2, 2)$ and $(2, 2, 3)$.
Write the line as a parametic equation and as $\ell(t) = \mathbf{a} + t\mathbf{v}$.

Parametric Form

$$\begin{cases} x = 1 + (2-1)t = 1+t \\ y = 2 + (2-2)t = 2 \\ z = 2 + (3-2)t = 2+t \end{cases}$$

Point-Direction Form

$$\ell(t) = a + tv$$
$$= (1,2,2) + t(1,0,1)$$

These are VERY similar and it is helpful to be able to use both and to be able to convert between them.

Inner Product and Length (M&T p. 21)

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. The inner product (or dot product) of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = a^T b = \sum_{k=1}^{n} a_k b_k$$

$\mathbb{R}^n$

The length (or magnitude) of $\mathbf{a}$ is:

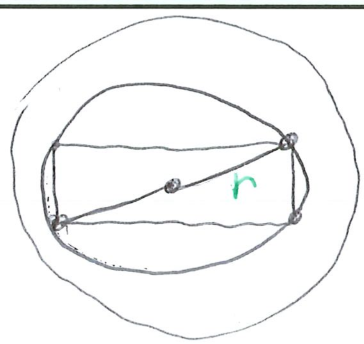
$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{k=1}^{n} a_k a_k} = \sqrt{\sum_{k=1}^{n} a_k^2}$$

Activity: Micro-Assignment (5 min)

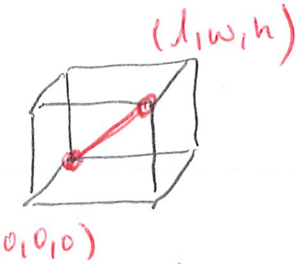
Suppose that a box has dimensions $(\ell, w, h) \in \mathbb{R}^3$

No name question etc.

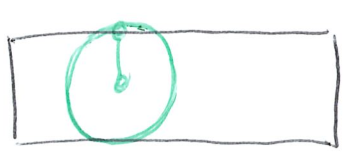
- What is the radius of the smallest sphere that could contain it?
- What is the radius of the largest sphere that it contains?



$$r = \frac{1}{2}\|(l, w, h)\|$$
$$= \frac{1}{2}\sqrt{l^2 + w^2 + h^2}$$



Longest distance inside the rectangular prism.

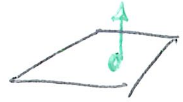


$$r = \frac{1}{2}\min(l, w, h)$$

= one half the minimum of l, w, and h.

Break to Resume at 13:10.

$\min(2, 1, 3) = 1$



Normal is at a 90° angle to plane.

Point-Normal Form of Plane (M&T p. 41)

The equation of a plane $\mathcal{P}$ through $\mathbf{x}_0 = (x_0, y_0, z_0)$ that has normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0 = n \cdot x - n \cdot x_0$$

Remark: We can re-arrange this equation to get: $Ax + By + Cz = \mathbf{n} \cdot \mathbf{x} = D$ for some constant D.

Theorem: Normalization

If $\mathbf{a}$ is a non-zero vector then $\mathbf{n} = \mathbf{a}/\|\mathbf{a}\|$ is a vector of unit length in the same direction as $\mathbf{a}$. We call $\mathbf{n}$ a unit vector because it has unit length. $\|n\| = 1$

$$\|n\| = \left\| \frac{a}{\|a\|} \right\| = \frac{\|a\|}{\|a\|} = 1$$

Example: Normalize a Vector

Give all unit vectors $\mathbf{n}$ perpendicular to the plane $(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot (x, y, z) = 2$.

There are only TWO unit normals perpendicular to a given plane.

$$n = \frac{i + 2j + 2k}{\|i + 2j + 2k\|}$$

$$= \frac{i + 2j + 2k}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{i + 2j + 2k}{\sqrt{9}} = \frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k$$

We also get: $n = -\left(\frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k\right)$

$$= -\frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k$$

Theorem: Inner Product and Angles (M&T p. 22)

If **a** and **b** are two non-zero vectors in $\mathbb{R}^n$ and $0 \leq \theta \leq \pi$ is the angle between **a** and **b** then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Remark: This is very cool because it works in *any* dimension.

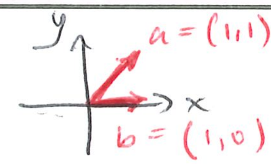
This works in $\mathbb{R}^{1000}$ even though we cannot visualize it!

Example: Find an Angle

Find the angle between the following sets of vectors.

- $\mathbf{a} = (1, 1)$ and $\mathbf{b} = (1, 0)$.
- $\mathbf{a} = (1, 1, 1, 1)$ and $\mathbf{b} = (1, \frac{1}{4}, -\frac{1}{4}, -1)$.

Let $a = (1,1)$ and $b = (1,0)$.



We get: $a \cdot b = 1 \cdot 1 + 1 \cdot 0 = 1$

$\|a\| \|b\| \cos\theta = \sqrt{1^2+1^2}\sqrt{1^2+0^2}\cos\theta = \sqrt{2}\cos\theta$

This gives $1 = \sqrt{2}\cos\theta \iff \frac{1}{\sqrt{2}} = \cos\theta$

$\iff \theta = \frac{\pi}{4}$

$a = (1,1,1,1) \qquad b = (1, \frac{1}{4}, -\frac{1}{4}, -1)$

We get: $a \cdot b = 1 \cdot 1 + 1 \cdot \frac{1}{4} + 1 \cdot (-\frac{1}{4}) + 1 \cdot (-1) = 0$

Thus, $0 = \|a\| \|b\| \cos\theta \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$

90° $\frac{\pi}{2}$ perpendicular orthogonal

Orthogonality (M&T p. 24)

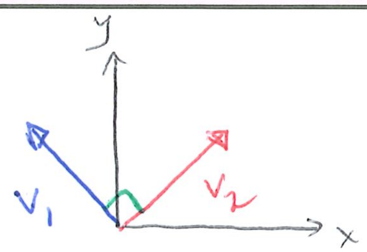
Suppose **a** and **b** are two non-zero vectors and $0 \leq \theta \leq \pi$ is the angle between them.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$$

In this case, we say that **a** and **b** are orthogonal. By convention, **0** is orthogonal to everything.

Example: An Orthogonal Frame

Check that the vectors $\mathbf{v}_1(\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ and $\mathbf{v}_2(\theta) = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are orthogonal for all values of θ. Sketch them in the plane.



Another nice thing to note:

$$\|v_1(\theta)\| = \|v_2(\theta)\| = 1.$$

For $v_1(\theta)$

$$\|v_1(\theta)\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$$

We check orthogonality:

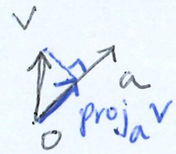
$$v_1 \cdot v_2 = (\cos\theta\, i + \sin\theta\, j)(-\sin\theta\, i + \cos\theta\, j)$$

$$= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

Thus, they are always orthogonal.

Orthogonal Projections (M&T p. 35-37)

The orthogonal projection of $\mathbf{v}$ on to $\mathbf{a}$ is:



$$\mathbf{p} = \text{proj}_{\mathbf{a}}\, \mathbf{v} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$$

The interpretation of $\mathbf{p}$ is as follows: Consider extending $\mathbf{a}$ to a line ℓ through the origin. The projection of $\mathbf{v}$ on to $\mathbf{a}$ is the vector on ℓ forming a right triangle with hypotenuse $\mathbf{v}$.

Example: Projecting onto Oneself

Suppose that $\mathbf{a}$ is non-zero. What is $\text{proj}_{\mathbf{a}}\, \mathbf{a}$?
Let $\lambda \neq 0$, what is $\text{proj}_{\mathbf{a}}\, \lambda\mathbf{a}$?

$$\text{proj}_a a = \frac{a \cdot a}{a \cdot a} a = \frac{\|a\|^2}{\|a\|^2} a = 1a = a$$

We need a non-zero for this to be defined.

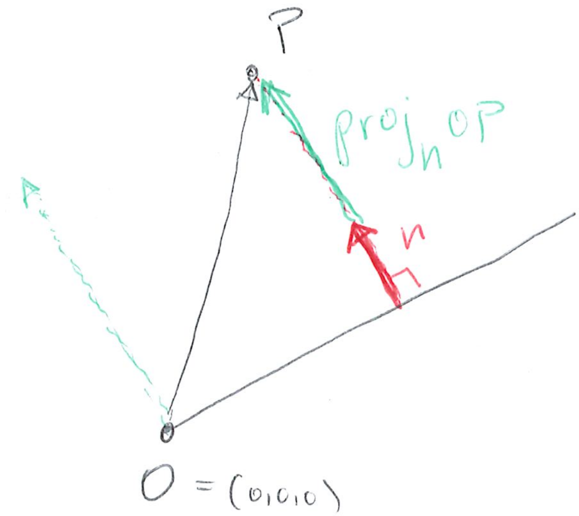
$$\text{proj}_a \lambda a = \frac{a \cdot (\lambda a)}{a \cdot a} a = \frac{\lambda (a \cdot a)}{a \cdot a} a = \lambda a$$

We use $a \cdot (kb) = k(a \cdot b)$ at this step.

Example: Distance to a Plane

Consider the point $P = (1,2,3)$ and the plane $(1,1,1)\cdot(x,y,z) = 0$.
Find the distance from P to the plane by projecting P on to the normal of the plane.

To measure the distance to P we measure $\|\text{proj}_n OP\|$.



$$\text{proj}_n OP$$
$$= \text{proj}_{(1,1,1)}(1,2,3)$$
$$= \frac{(1,1,1)\cdot(1,2,3)}{(1,1,1)\cdot(1,1,1)}(1,1,1)$$
$$= \frac{1+2+3}{3}(1,1,1) = 2(1,1,1)$$

We get: the distance is
$$\|\text{proj}_n OP\| = 2\|(1,1,1)\|$$
$$= 2\sqrt{1^2+1^2+1^2}$$
$$= 2\sqrt{3}$$

$$\|kv\| = |k|\,\|v\|$$

Matrices and Invertibility (M&T p. 66)

The identity matrix $I = I_n$ is $n \times n$ matrix such that: $AI = IA = A$. For 3×3 matrices it is:

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

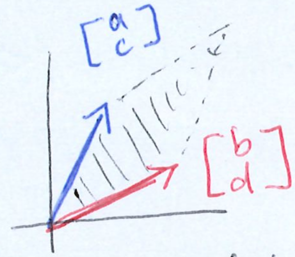
An $n \times n$ matrix A is invertible if there is another $n \times n$ matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

Determinants (M&T p. 32)

The determinant of a 2×2 matrix is:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$



The area of this parallelogram is $\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|$

In general, for an $n \times n$ matrix A:

?

$$\det A = \sum_{k=1}^{n} (-1)^{i+j} \det(A_{ij})$$

a_{ij}

where A_{ij} is the matrix A without row i and column j.

Example: Find a Determinant

Compute the determinant of the following matrix.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

$$\det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{pmatrix} = 1 \cdot \det\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 0 & 0 \\ 4 & 5 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$$

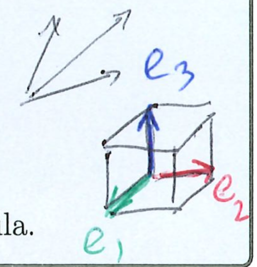
$$= 1 \cdot 3 \cdot 5 - 0 + 2 \cdot (-3 \cdot 4) = 15 - 24 = -9$$

Important Properties of The Determinant (M&T p. 32-33)

1. ☆ $\det([\mathbf{v}_1 \ldots \mathbf{v}_n])$ = the signed volume of the parallelpiped spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
2. $\det(A) \neq 0 \iff A$ is invertible.
3. $\det(AB) = \det(A)\det(B)$
4. $\det(I) = 1$

n-dimensional parallelogram

A parallelpiped has edges $Ae_1 \; Ae_2 \cdots Ae_n$



Note: Property (1) will become more-and-more imporant as the course goes on.
It is central to the material in Week 11 when we study the change-of-variable formula.

Geometry of the 2×2 Determinant (M&T p. 39)

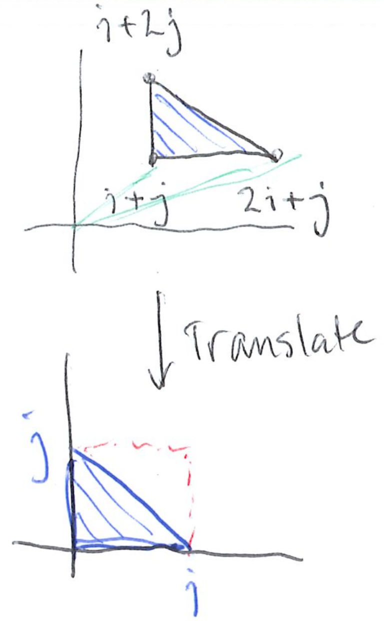
Suppose that A is a 2×2 matrix $[\mathbf{a}\ \mathbf{b}]$ with columns $\mathbf{a}$ and $\mathbf{b}$.
The absolute value of $\det(A)$ is the area of the parallelogram whose adjacent sides are $\mathbf{a}$ and $\mathbf{b}$.

Example: Find an Area

Find the area of the triangle with vertices $\mathbf{i}+\mathbf{j}$, $2\mathbf{i}+\mathbf{j}$, and $\mathbf{i}+2\mathbf{j}$ using determinants.
Use geometry to check your work.

Shift the triangle to the origin.

$$(2i+j)-(i+j) = i$$
$$(i+2j)-(i+j) = j$$



Apply determinant

$$\det(i\ j) = \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1$$

To get the area of the triangle, we must divide by two.

$$A = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$A = \frac{1}{2}bh = \frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2}$$

Yay! They match.

The Cross Product (M&T p. 35)

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ then: very weird use of notation.

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \mathbf{i} \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

Example: Compute a Cross Product

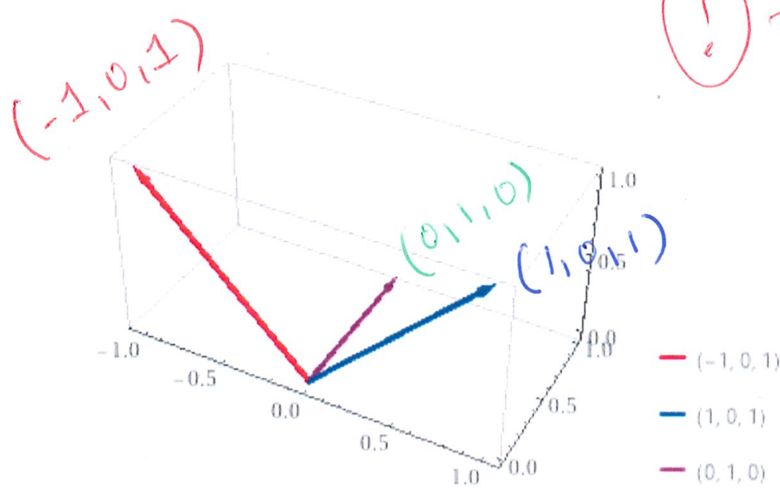
Compute the cross product of $(1, 0, 1) \times (0, 1, 0)$.

$$(1,0,1) \times (0,1,0) = \det \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= i \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (-1, 0, 1)$$

Can you expand along any row/column?

(!) The cross product is orthogonal to its inputs.



Example: Crossing with Oneself

Suppose that $\mathbf{a}$ is non-zero. What is $\mathbf{a} \times \mathbf{a}$? Let $\lambda \neq 0$, what is $\mathbf{a} \times (\lambda \mathbf{a})$?

$$a \times a = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0i + 0j + 0k.$$

$$a \times (\lambda a) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \lambda a_1 & \lambda a_2 & \lambda a_3 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \lambda \cdot 0 = 0$$

Important Properties of Cross-Product (M&T p. 37)

- The length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
- $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$
- ☆ $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Example: Find the Normal of a Plane

Suppose that a plane through the origin contains the vectors $\mathbf{v}_1 = (1, 2, 3)$ and $\mathbf{v}_2 = (2, 1, 0)$. Find a unit normal to the plane, and write it in the form $\mathbf{n} \cdot \mathbf{x} = 0$.

We find a normal to the plane.

$$V = V_1 \times V_2$$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{vmatrix} = (-3, +6, -3)$$

We form a unit normal n.

$$n = \frac{V}{\|V\|} = \frac{(-3, 6, -3)}{\sqrt{3^2 + 6^2 + 3^2}} = \frac{(-3, 6, -3)}{\sqrt{54}}$$

$$= \frac{(-3, 6, -3)}{3\sqrt{6}} = \frac{1}{\sqrt{6}}(-1, 2, -1)$$

Thus, our plane is:

$$\frac{1}{\sqrt{6}}(-1, 2, -1) \cdot (x, y, z) = 0$$

Example: The Triple Product Identity

Given three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ we have:

LHS RHS

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$$

We prove the equality by computing LHS and RHS.

The LHS is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \det \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \end{array}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \left(b_2c_3 - c_2b_3, -(b_1c_3 - c_1b_3), (b_1c_2 - c_1b_2) \right)$$

$$= a_1b_2c_3 - a_1c_2b_3 - a_2b_1c_3 + a_2c_1b_3 + a_3b_1c_2 - a_3c_1b_2$$

The RHS is:

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$