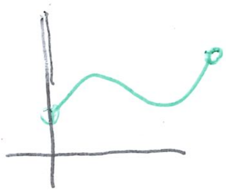
Week 3: Geometry, Limits, and Derivatives

Visualization Technique #1 : Graphs of Functions (M&T p. 77)

If $f : \mathbb{R} \to \mathbb{R}$ then the graph of $y = f(x)$ is: graph $f = \{(x, y) : y = f(x)\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.
In general, the graph of $f : \mathbb{R}^n \to \mathbb{R}$ is:

$$\text{graph } f = \{(\mathbf{x}, x_{n+1}) : x_{n+1} = f(\mathbf{x})\} \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$$

often in $\mathbb{R}^3$ or more.



$y = f(x)$

$\text{graph}(f) \subseteq \mathbb{R}^2$
$= \{(x, y) : y = f(x)\}$

The function is $f: \mathbb{R} \to \mathbb{R}$ but its graph lives in two-dimensional space.

(!) Humans are very two-dimensional. Our eyes see two-dimensional pictures.

It is very hard to draw graphs of $f: \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$.

Activity: What do you notice?

The picture below shows Horsethief Canyon outside of Drumheller Alberta.

- Is anyone in the class from Alberta?
- Has anyone in the class visited Horsethief Canyon?
- What do you notice about this picture?

This activity was inspired a conversation with Mike Pawliuk (UTM).

- Levels of different kinds of rocks.
- ~~Different~~ Same rocks are all at equal heights
- Hard to distinguish foreground vs background.

Visualization Technique #2: Level Sets (M&T p. 79)

If $f : \mathbb{R}^n \to \mathbb{R}$ then the level set of value c:

$$L_c = \{\mathbf{x} : c = f(\mathbf{x})\} \subset \mathbb{R}^n$$ = all the points of the domain that output the value c.

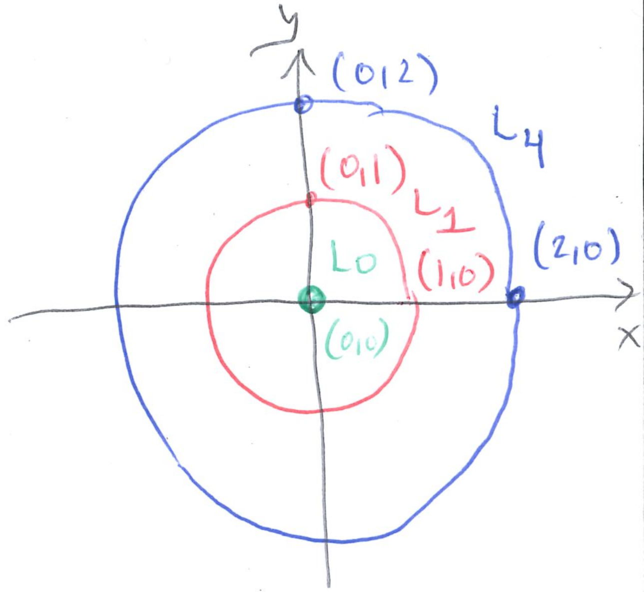
If $n = 2$ then this set is a level curve (or level contour). If $n = 3$ then this set is a level surface.

Example: Level Sets of a Paraboloid

Sketch the level sets L_c for $c = -1, 0, 1, 4$ of the function $f(x, y) = x^2 + y^2$.

Sketch $L_4 = \{\vec{x} : f(\vec{x}) = 4\}$

$x^2 + y^2 = 4$



Sketch $L_1 = \{\vec{x} : f(\vec{x}) = 1\}$

$x^2 + y^2 = 1$

Sketch $L_0 = \{\vec{x} : f(\vec{x}) = 0\}$

$x^2 + y^2 = 0$

Sketch $L_{-1} = \{\vec{x} : f(\vec{x}) = -1\}$

$x^2 + y^2 = -1$ ← No solutions! This level curve is empty.

This is the process that people use to draw TOPOGRAPHIC MAPS.

It lets us visualize graphs $f : \mathbb{R}^2 \to \mathbb{R}$ as two-dimensional pictures.

Example: Complicated Level Sets

Sketch the level sets L_c for $c = -2, 2$ of the following function.

$$f(x,y) = \frac{x^2+y^2}{xy}$$

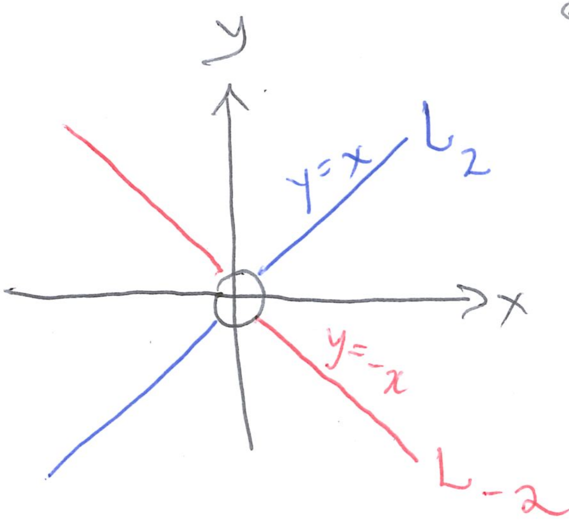
Write the level set condition. for $c=2$.

$$2 = \frac{x^2+y^2}{xy} \iff 2xy = x^2+y^2$$
$$\iff x^2-2xy+y^2=0$$
$$\iff (x-y)^2=0$$
$$\iff x-y=0 \iff y=x$$

This is only defined when x and y are non-zero.

Write the level set condition for $c=-2$.

$$-2 = \frac{x^2+y^2}{xy} \iff x^2+2xy+y^2=0$$
$$\iff (x+y)^2=0$$
$$\iff y=-x$$ when x and y are both non-zero.

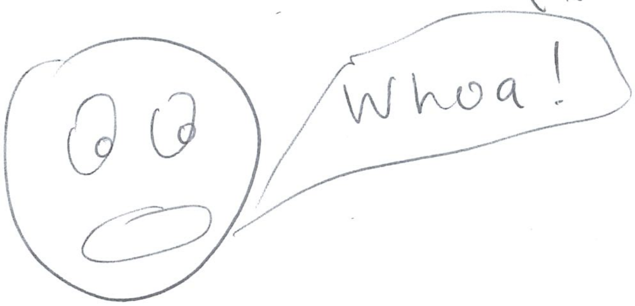


The origin is NOT included on the level sets L_{-2} or L_2.

Example: A Complicated Surface (M&T p. 84)

Use CalcPlot3D to sketch the surface $f(x, y) = (x^2 + 3y^2) \exp(1 - x^2 - y^2)$.

$$= (x^2 + 3y^2) e^{1-x^2-y^2}$$



Activity: Use a 3D Graphing Calculator

Use CalcPlot3D to visualize some stuff. It is a very powerful tool.

- Plot graphs.
- Plot contours.

For inspiration, check out these famous surfaces.

Some Famous Surfaces (M&T p. 82-83)

Spheres $x^2 + y^2 + z^2 = c^2$

Paraboloid $z = x^2 + y^2$ $\Longleftrightarrow f(x,y) = x^2 + y^2$

Saddle $z = x^2 - y^2$ $\Longleftrightarrow g(x,y) = x^2 - y^2$

Hyperboloid $x^2 + y^2 - z^2 = c$

Difficult or tricky material.

The following material is a bit more proof-y than the rest of the course. We are including it to give you a glimpse in to MAT B43 and C27.

Balls and Open Sets (MAT C27 Topology Sneak-Peek) (M&T p. 88)

An open ball centered at $\mathbf{p}$ of radius $\epsilon > 0$ is the set:

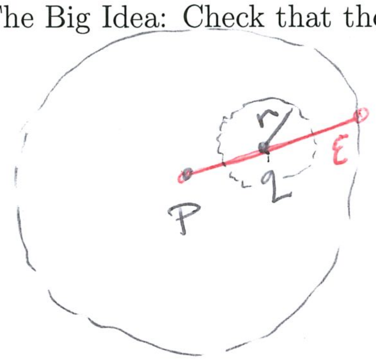
$$D_\epsilon(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{p}\| < \epsilon\}$$

We say that $U \subset \mathbb{R}^n$ is an open set when: for all $\mathbf{p} \in U$ there is $\epsilon > 0$ such that: $D_\epsilon(\mathbf{p}) \subseteq U$. Alternatively, we say that U is a neighbourhood of $\mathbf{p}$. "An open set contains an open ball around each of its points."

Example: Is an open ball also an open set?

We should check that an "open ball" is an "open set".

The Big Idea: Check that there is a radius $r > 0$ such that $D_r(\mathbf{q}) \subset D_\epsilon(\mathbf{p})$ for every $\mathbf{q} \in D_\epsilon(\mathbf{p})$.



$D_\varepsilon(p)$

For any point $q \in D_\varepsilon(p)$ we need a radius $r > 0$ such that:

$$D_r(q) \subseteq D_\varepsilon(p)$$

We pick $r = \varepsilon - \|p - q\|$

This will be positive because $\|p-q\| < \varepsilon$.

We now check that $D_r(q) \subset D_\varepsilon(p)$.

We pick $x \in D_r(q)$ and check $x \in D_\varepsilon(p)$.

$$\|x - p\| = \|x - q + q - p\|$$

$\leq \|x - q\| + \|q - p\|$ by the triangle inequality

$$< r + \|q - p\| = (\varepsilon - \|p - q\|) + \|p - q\|$$

$= \varepsilon$ Yay! $x \in D_\varepsilon(p)$:)

Thus, $D_r(q) \subset D_\varepsilon(p)$.

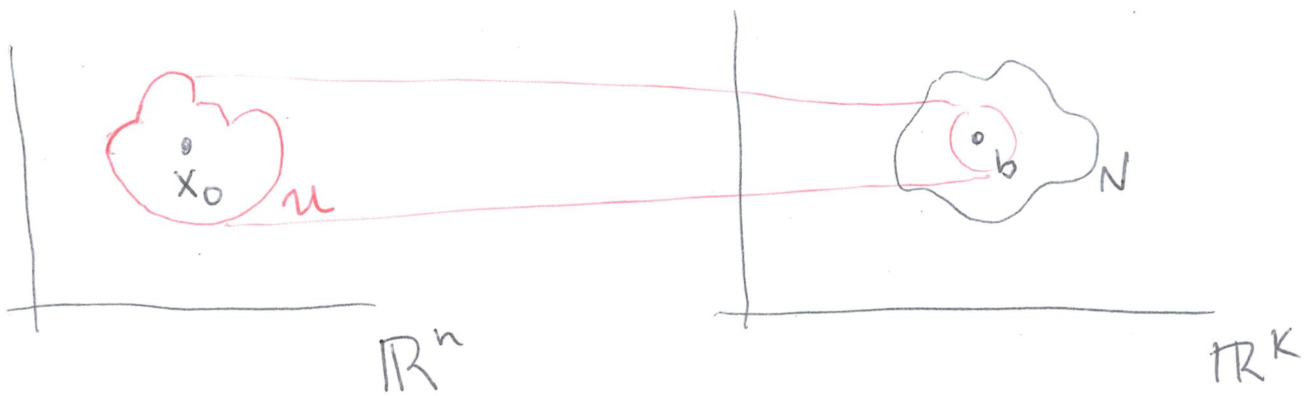
Limits (Topology Sneak-Peek) (M&T p. 92)

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^k$. We say that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$$

if (for every neighbourhood N of $\mathbf{b}$)(there is a neighbourhood U of $\mathbf{x}$) such that:

$$\mathbf{x} \in U \Longrightarrow f(\mathbf{x}) \in N$$



For any neighbourhood of b
There is a neighbourhood of x_0
such that $f(\mathcal{U}) \subseteq N$.

(!) Very strong condition.

It suffices to take $\mathcal{U}$ and N to be balls.

$\epsilon\delta$-Style Limits (M&T p. 92)

If we want to avoid talking about balls and neigbourhoods, then we can re-define limits in an $\epsilon\delta$-style.

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} \iff (\forall_{\epsilon>0}\exists_{\delta>0} : 0< \|\mathbf{x}-\mathbf{x}_0\| < \delta \implies \|f(\mathbf{x})-\mathbf{b}\| < \epsilon)$$

Example: A Rigorous Limit in Two Dimensions

Prove that $\lim_{(x,y)\to(2,1)} x+2y = 4$. Of course! $2\cdot 1 + 2\cdot 1 = 2+2 = 4$

For all $\varepsilon > 0$ we need $\delta > 0$ such that:

$$\|(2,1)-(x,y)\| < \delta \implies \|(x+2y)-4\| < \varepsilon.$$

We investigate how to pick δ.

$$\|x+2y-2-2\cdot 1\| = \|(x-2)+2(y-1)\|$$

$$\le \underbrace{\|x-2\|}_{<\frac{\varepsilon}{3}} + 2\underbrace{\|y-1\|}_{<\frac{\varepsilon}{3}}$$

we should pick: $\delta < \frac{\varepsilon}{3}$.

Want: $< \frac{\varepsilon}{3}$, $< \frac{\varepsilon}{3}$

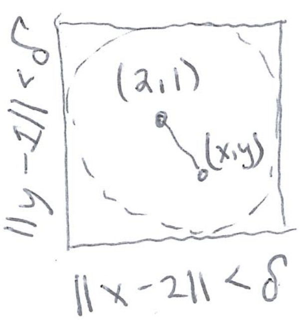
$$< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \frac{3\cdot\varepsilon}{3} = \varepsilon$$

given $\varepsilon > 0$ we pick $\delta < \frac{\varepsilon}{3}$

$$\|(x+2y)-4\| \le \|x-2\| + 2\|y-1\| \text{ by triangle ineq.}$$

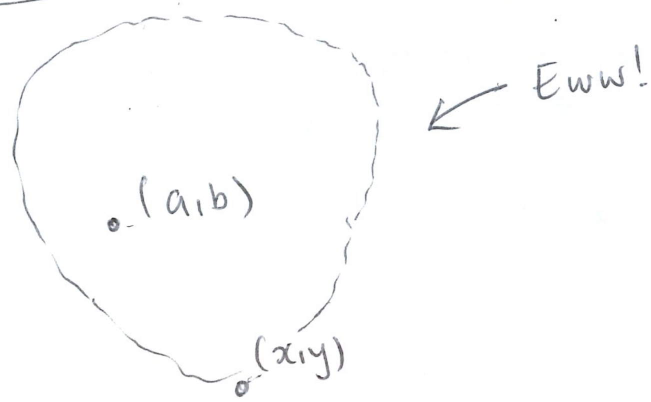
$$\le \delta + 2\delta = 3\delta \text{ by Ball-Box inequality.}$$

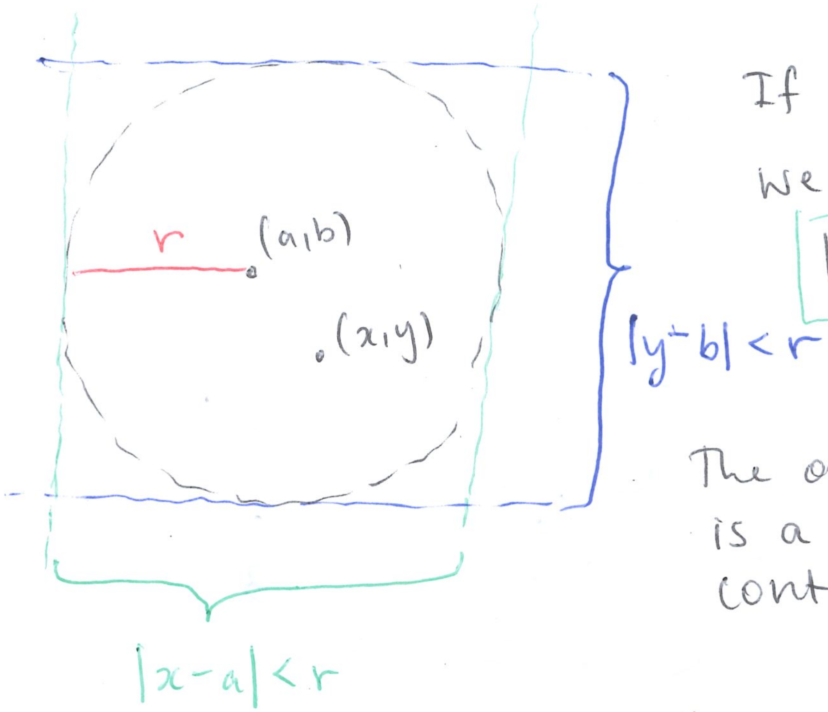
$$< 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$



See reverse.

Ball-Box Inequality





If we know $\|(x,y)-(a,b)\| < r$ we want to argue that $\boxed{|x-a|}$ and $\boxed{|y-b| < r}$

The overlap of these regions is a square that completely contains the ball.

Being inside the ball $\|(x,y)-(a,b)\| < r$ implies that you are in the box

$|x-a| < r$ and $|y-b| < r$.

Properties of Limits (M&T p. 95)

1. Limits are unique.

$$\left(\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1 \quad \& \quad \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2\right) \implies \mathbf{b}_1 = \mathbf{b}_2$$

2. Limits are determined by their component functions. Suppose $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$.

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_k) \iff \left(\lim_{\mathbf{x}\to\mathbf{x}_0} f_1(\mathbf{x}) = \mathbf{b}_1 \quad \& \cdots \& \quad \lim_{\mathbf{x}\to\mathbf{x}_0} f_k(\mathbf{x}) = \mathbf{b}_k\right)$$

3. Limits respect addition. $f: \mathbb{R}^n \to \mathbb{R}^k \Leftrightarrow f = (f_1, f_2 \ldots f_k)$

$$\boxed{\lim_{x\to x_0}(f+g) = \lim_{x\to x_0} f + \lim_{x\to x_0} g}$$

4. Limits respect scaling.

$$\boxed{\lim_{x\to x_0}\left(k f(x)\right) = k \lim_{x\to x_0} f(x) \quad \text{k is a scalar here}}$$

x is a vector here.

5. Limits respect multiplication.

$$\boxed{\lim_{x\to x_0}(fg) = \left(\lim_{x\to x_0} f\right)\left(\lim_{x\to x_0} g\right).}$$

$f: \mathbb{R}^n \to \mathbb{R}$
$g: \mathbb{R}^n \to \mathbb{R}$

Activity: Think-Pair-Board (3 min)

Think about these following task alone for one minute, then chat with your neighbour about it for two minutes, then share your ideas with the class by writing them on the board.

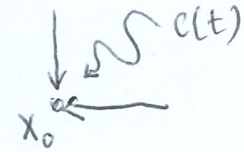
Carefully write out the statements of Properties 3-5 of Limits.
They should be direct generalizations of the limit laws for single variable functions.

$[a,b] \to \mathbb{R}^2$

Limits along Paths (M&T p. 97)

A function $\mathbf{c} : [a, b] \to \mathbb{R}^n$ is a <u>curve</u> or <u>path</u>. If $\lim_{t\to b} \mathbf{c}(t) = \mathbf{x}_0$ then we can compute the limit

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$$



along the path $\mathbf{c}(t)$ as:

$$\lim_{t\to b} f(\mathbf{c}(t)) = \text{"} f(x_0) \text{"}$$

Note: The value of this limits depends on the choice of path.

Theorem: Existence implies equality

If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ exists then it agrees along all paths. Formally, if $\mathbf{c}(t)$ is a path with $\lim_{t\to t_0} c(t) = \mathbf{x}_0$ then:

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \lim_{t\to t_0} f(\mathbf{c}(t))$$

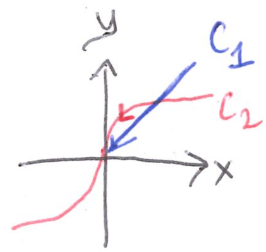
! If two paths do not agree then the limit does NOT exist.

Note: This theorem is very helpful for showing limits do not exist.

Example: Choice of Path Matters!

Consider the following function.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



1. Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $\mathbf{c}_1(t) = (t,t)$.
2. Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $\mathbf{c}_2(t) = (t^3,t)$.

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

① $\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $c_1(t) = (t,t)$

$$\lim_{t\to0} f(c_1(t)) = \lim_{t\to0} f(t,t) = \lim_{t\to0} \frac{t\cdot t^3}{t^2+t^6} = \lim_{t\to0} \frac{t^2}{1+t^4} = \frac{0}{1+0^4} = 0$$

② Along $c_2(t)$

$$\lim_{t\to0} f(c_2(t)) = \lim_{t\to0} f(t^3,t) = \lim_{t\to0} \frac{t^3\cdot t^3}{(t^3)^2+t^6} = \lim_{t\to0} \frac{t^6}{t^6+t^6} = \lim_{t\to0} \frac{1}{2} = \frac{1}{2}$$

For every point on $c_2(t)$, $f(x,y)$ outputs $z = \frac{1}{2}$

Continuity (M&T p. 97)

We say that $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous at $\mathbf{x}_0$ if and only if:

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$$

"Limit agrees with the output value."

A function is continuous if it is continuous at every point of its domain.

Example: A Discontinuous Function

Consider the following function.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is $f(x,y)$ continuous at $(x,y) = (0,0)$? Graph it in CalcPlot3D and investigate.

Hard No!

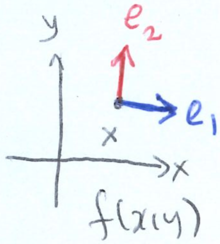
The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Therefore $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$

Partial Derivatives (M&T p. 106)

Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function.
The partial derivatives of f are the functions:

$$\frac{\partial f}{\partial x_k}(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_k) - f(\mathbf{x})}{h}$$



This derivative measures the rate of change of f in the direction x_k.
There are different notations for this derivative:

$$\frac{\partial f}{\partial x_k}(\mathbf{x}) \quad \text{or} \quad \left.\frac{\partial f}{\partial x_k}\right|_{\mathbf{x}} \quad \text{or} \quad f_{x_k}(\mathbf{x})$$

Activity: Mini-Assignment (5 min)

Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x, y) = \|(x, y)\|^2$ at $(x, y) = (1, 2)$.

Notice: $f(x,y) = \|(x,y)\|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

$\frac{\partial f}{\partial y} = 4$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f((1,2)+h(1,0)) - f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h, 2) - f(1,2)}{h} = \lim_{h\to 0} \frac{(1+h)^2 + \cancel{2^2} - (1^2 + \cancel{2^2})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} + 2\cancel{h} + h^2 - \cancel{1^2}}{\cancel{h}} = \lim_{h\to 0} 2 + h = \cancel{\$} \; 2 + 0 = 2$$

C^1-functions (M&T p. 114)

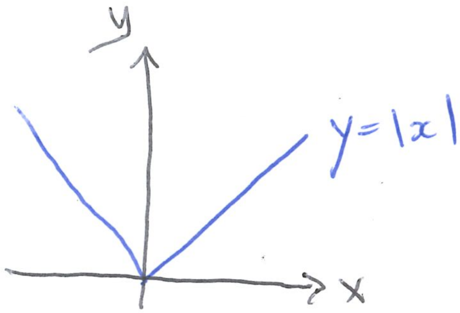
A function f belongs to class C^1 (or is C^1) if all its partial derivatives exist and are continuous. Inductively, we define the following: A function f belongs to class C^k (or is C^k) if all its partial derivatives are C^{k-1}.

The higher the value of k, the nicer the function.
The "niceness" here is number of derivatives.

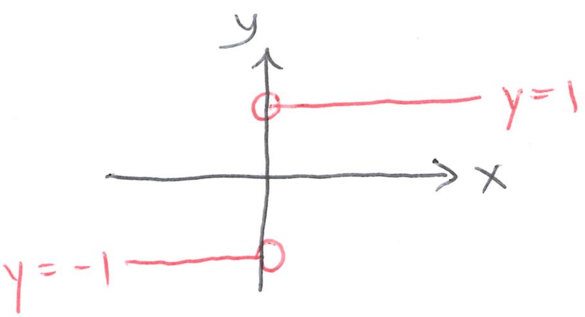
Being C^k (especially C^2) is a technical criteria that we will often need for important theorems.

Example: Absolute Value is Not C^1

$f(x) = |x|$ is continuous but not C^1.



This function is beautiful.
It is symmetric.
It is continuous.



$$\frac{dy}{dx} = \frac{d|x|}{dx} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

This is a mess!
It is not defined everywhere.
It is NOT continuous.

Example: The Fast Way to Compute Partial Derivatives

Compute the partial derivatives of $f(x, y) = xy\exp(1 - x^2 - y^2)$ as follows:

- Assume y is constant and compute f_x.
- Assume x is constant and compute f_y.

$$f(x,y) = xye^{1-x^2-y^2}$$

Calculate f_x

Assume y is constant.

$$f_x = \frac{\partial}{\partial x}\left[xy\,e^{1-x^2-y^2}\right]$$

$$= ye^{1-x^2-y^2} + xy\frac{\partial}{\partial x}\left[e^{1-x^2-y^2}\right] \quad \text{\# Product rule}$$

$$= ye^{1-x^2-y^2} + xy(-2x)e^{1-x^2-y^2} \quad \text{\# Chain rule}$$

$$= (y - 2x^2y)e^{1-x^2-y^2}$$

(!) Notice $\frac{\partial}{\partial x}[xy] = y$ because y is constant

calculate f_y

Assume x is constant.

$$f_y = \frac{\partial}{\partial y}\left[xye^{1-x^2-y^2}\right]$$

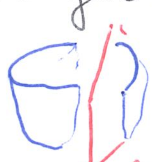
$$= xe^{1-x^2-y^2} + \frac{\partial}{\partial y}\left[e^{1-x^2-y^2}\right](xy)$$

$$= xe^{1-x^2-y^2} + (-2y)e^{1-x^2-y^2}(xy)$$

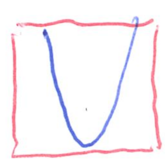
$$= (x - 2xy^2)e^{1-x^2-y^2}$$

What is going on geometrically?

$z = f(x,y)$



cut along a plane



Calculate the derivate of the curve in the plane.

Linear Approximation (M&T p. 109)

The linear approximation of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at (x_0, y_0) is

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

This is also the equation of the tangent plane to $f(x, y)$ at (x_0, y_0).

Activity: Think-Pair-Share (3 min)

Think about these questions alone for one minute, then chat with your neighbour about them for two minutes, then share your ideas with the class.

1. What properties of tangent lines to $f : \mathbb{R} \to \mathbb{R}$ does this generalize?
2. Why is this the correct definition of a tangent plane?

Solo

Example: Find a Tangent Plane

Find the tangent plane to $z = x^2 + y^2$ at $(x, y, z) = (1, 1, 2)$.

Solo

Functions $f : \mathbb{R}^n \to \mathbb{R}^k$ (M&T p. 111)

A function $f : \mathbb{R}^n \to \mathbb{R}^k$ has n inputs and k outputs.

$$f(\mathbf{x}) = f(x_1, \ldots, x_n) = (f_1, \ldots, f_k)$$

Note: Each of its k outputs is a function $f_i : \mathbb{R}^n \to \mathbb{R}$. "the component functions of f"

Derivatives (M&T p. 111)

If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a function $f = (f_1, f_2, \ldots, f_k)$ then the <u>derivative</u> $\mathbf{D}f(\mathbf{x}_0)$ is the matrix:

$$\mathbf{D}f = \left[\frac{\partial f_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix}$$

This is the matrix of partial derivatives of f at the $\mathbf{x}_0$. Note that this is a $\boxed{k \times n \text{ matrix}}$.

Example: A Function from the Plane to the Plane

Find the derivative of $f(x, y) = (xy, x + y)$ $\qquad f : \mathbb{R}^2 \to \mathbb{R}^2$

These are functions
$f : \mathbb{R}^2 \to \mathbb{R}^1$

$$Df = \begin{bmatrix} \frac{\partial(xy)}{\partial x} & \frac{\partial(xy)}{\partial y} \\ \frac{\partial(x+y)}{\partial x} & \frac{\partial(x+y)}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

Big deal in machine learning.
"gradient ascent"

Gradients (M&T p. 112)

An important special case of the derivative is the gradient of $f : \mathbb{R}^n \to \mathbb{R}$.

∇ = "nabla" = "grad"

$$\nabla f = \mathbf{D}f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right]$$

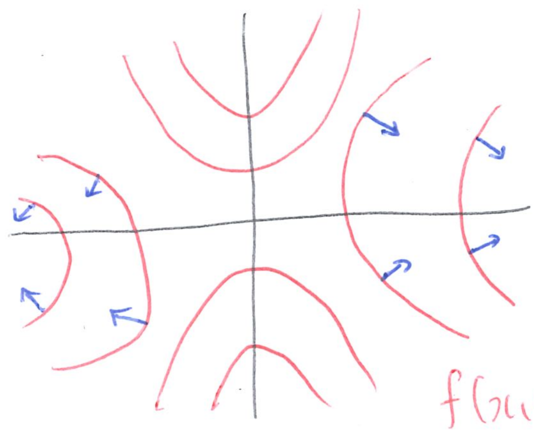
Notice: This outputs a vector for each point in the plane. This is a vector field.

f function $\to$ ∇f vectors

Activity: Sketch a Gradient Vector Field

Calculate the gradient ∇f for $f(x, y) = x^2 - y^2$. Use CalcPlot3D to sketch the gradient vector field.

$$\nabla f = \left[\frac{\partial(x^2-y^2)}{\partial x}, \frac{\partial(x^2-y^2)}{\partial y}\right] = [2x, -2y]$$



$f(x,y) = x^2 - y^2$

Notice:
- ∇f is a vector field.
- $f : \mathbb{R}^2 \to \mathbb{R} \implies \nabla f : \mathbb{R}^2 \to \mathbb{R}^2$
- ∇f perpendicular to level sets.

Differentibility (M&T p. 111)

Let $\mathbf{D} = \mathbf{D}f(\mathbf{x}_0)$. We say that f is differentiable at x_0 if the following limit exists:

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0$$

Note: The product $\mathbf{D}(\mathbf{x}-\mathbf{x}_0)$ is a matrix $\mathbf{D}$ times the column vector $\mathbf{x}-\mathbf{x}_0$.

Example: A Linear Function is Differentiable

Check that the function $f(x, y) = (x + 2y, 3x + 4y)$ is differentiable at $(x, y) = (1, 0)$.

Solo

Theorem: Linear Approximation

If f is differentiable at $\mathbf{x}_0$ and $\mathbf{D} = \mathbf{D}f(\mathbf{x}_0)$ then $f(\mathbf{x}) \approx f(\mathbf{x}_0) + \mathbf{D}(\mathbf{x} - \mathbf{x}_0)$ near $\mathbf{x}_0$.

Example: A Linear Approximation

Find the linear approximation of $f(x, y) = (xe^y, ye^x)$ at $(x, y) = (1, 1)$.

Solo