Week 5: Higher Order Derivatives and Taylor's Theorem

Iterated Partial Derivatives (M&T p. 150)

If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable then we can form the following partial derivatives.

rate of change in the x-direction $\longrightarrow \dfrac{\partial f}{\partial x} \quad \dfrac{\partial f}{\partial y} \leftarrow$ rate of change in the y-direction

These derivatives are themselves functions $\mathbb{R}^2 \to \mathbb{R}$.
Therefore, we have the following second order partial derivatives.

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2}$$

Four types of second derivatives

In general, the kth order partial derivatives involve k differentiations.
We call these mixed partial derivatives or iterated partial derivatives.

Example: Calculate Some Mixed Partial Derivatives

Calculate the following mixed partial derivative.

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}[\sin(xyz)]\right]\right]$$

Work from inside outwards.

compute derivative in z-direction

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[xy\cos(xyz)\right]\right]$$

compute derivative in y-direction

$$= \frac{\partial}{\partial x}\left[x\cos(xyz) + xy\left(-\sin(xyz)xz\right)\right]$$

$$= \frac{\partial}{\partial x}\left[x\cos(xyz) - x^2yz\sin(xyz)\right]$$

compute derivative in x-direction

$$= \cos(xyz) - xyz\sin(xyz) - 2xyz\sin(xyz) - x^2y^2z^2\cos(xyz)$$

Recall: Class C^k (M&T p. 150)

Recall that f is C^1 if all its partial derivatives exist and are continuous.
We say that f belongs to class C^k (or is C^k) if all its derivatives of order $\leq k$ exist and are continuous.

☆ Theorem: Equality of Mixed Partials (M&T p. 151)

If $f(x, y)$ belongs to class C^2 then the mixed partial derivatives are equal.

Proof in B42

Question: Is this an equivalence?

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This theorem generalizes to kth order derivatives of C^k functions.

Example: Check the Equality of Mixed Partials

Check the equality of mixed partial derivatives by verifying:

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[1x^2 + 2xy + 3y^2\right]\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[1x^2 + 2xy + 3y^2\right]\right]$$

This works for all polynomials.

$$= \frac{\partial}{\partial x}\left[0 + 2x + 0\right]$$

$$= 0 + 2 + 0 = 2$$

$$= \frac{\partial}{\partial y}\left[1 \cdot 2 + 2y + 0\right]$$

$$= 0 + 2 + 0 = 2$$

Polynomials are in C^k for all k.

Yay! They match. :)

Subscript Notation for Derivatives (M&T p. 152)

An alternative notation for partial derivatives is the following.

$$f_{yx} = [f_y]_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

Notice that the order of x and y is reversed in the two notations! Luckily, for C^2 functions, $f_{xy} = f_{yx}$.

Activity: Calculate as a Class (3 min)

Consider the function $f(x, y, z) = x^2y + xy^2 + yz^2$.
Calculate f_{xy}, f_{yz}, and f_{xyz}. Write your final answers on the board.

$f_{xy} = 2x + 2y = f_{yx}$

$f_{yz} = 2z = f_{zy}$

$f_{xyz} = 0$

Partial Differential Equations (PDEs) are equations relating partial derivatives

LOTS of equations in physics are PDEs.

The Heat Equation (Sneak Peek to MAT B42) (M&T p. 154)

Let $\mathcal{B} \subset \mathbb{R}^2$ denote some physical body in two dimensional space. Suppose that $T(x, y, t)$ is the temperature at time t of the point $(x, y) \in \mathcal{B}$. The temperature must satisfy the heat equation:

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t}$$

B42: How do we solve the Heat Equation?

In this equation k is a constant which describes the heat conductivity of the body $\mathcal{B}$.

Example: A Solution of the Heat Equation

Check that $T(x, y, t) = e^{-kt}(\cos(x) + \cos(y))$ satisfies the heat equation.

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left[e^{-kt}\left(\cos(x)+\cos(y)\right)\right]$$

$$= -ke^{-kt}\left(\cos(x)+\cos(y)\right)$$

Yay! They match.

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = k\left(\frac{\partial^2}{\partial x^2}\left[e^{-kt}\left(\cos(x)+\cos(y)\right)\right] + \frac{\partial^2}{\partial y^2}\left[e^{-kt}\left(\cos(x)+\cos(y)\right)\right]\right)$$

$$= k\left(\frac{\partial}{\partial x}\left[e^{-kt}\left(-\sin(x)+0\right)\right] + \frac{\partial}{\partial y}\left[e^{-kt}\left(0+(-\sin(y))\right)\right]\right)$$

$$= k\left(-e^{-kt}\cos(x) - e^{-kt}\cos(y)\right)$$

$$= -ke^{-kt}\left(\cos(x)+\cos(y)\right)$$

The Wave Equation (Sneak Peek to MAT B42) (M&T p. 155)

Consider a string stretched along the x-axis. Its height at position x and time t is given by $u(x,t)$. If it is disturbed then it will vibrate according to the the wave equation:

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

The constant c is the rate at which vibrations travel along the string. We will usually pick $c = 1$.

Example: A Solution of the Wave Equation

Check that $u(x,t) = \sin(x)\sin(ct)$ is a solution of the wave equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[\sin(x)\sin(ct)\right] = \frac{\partial}{\partial x}\left[\cos(x)\sin(ct)\right] = -\sin(x)\sin(ct)$$

$$c^2\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2}{\partial t^2}\left[\sin(x)\sin(ct)\right] = c^2\frac{\partial}{\partial t}\left[\sin(x)\cos(ct)\cdot c\right] = c^2\left[-\sin(x)\sin(ct)\cdot c^2\right] = -c^4\sin(x)\sin(ct)$$

Oops! These c's should cancel out. Will check book for correct formula.

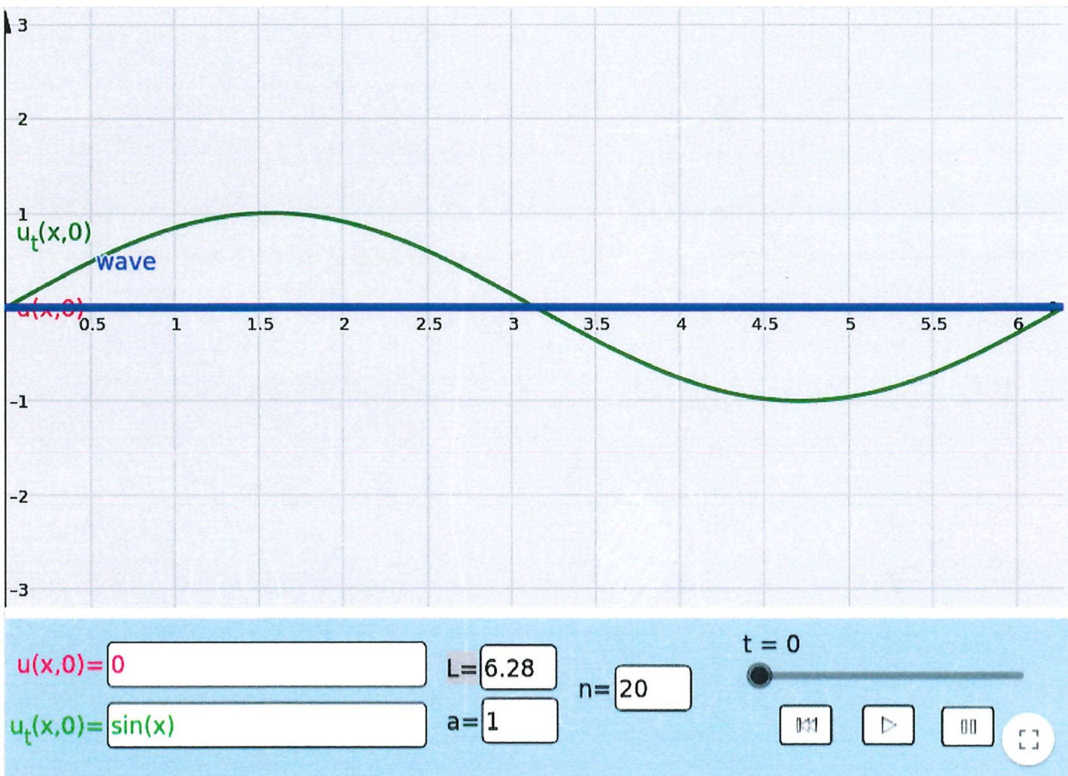
The c^2 should be with the space variables (x) and not the time variable (t).

Activity: Simulate a Wave (5 min)

Juan Carlos Ponce Campuzano (University of Queensland, Australia) has a wonderful simulator.

https://www.geogebra.org/m/eKkFV8uz

Play around with it! Check out various waves.
The solution $u(x,t) = \sin(x)\sin(ct)$ (with $c = 1$) corresponds to the following settings.



In the settings: $L = \text{Length} = 6.28 \approx 2\pi$ and $a = c = 1$.

Taylor's Theorem in One-Dimension (M&T p. 159)

The single variable Taylor's Theorem (roughly) says: $f(x) \rightsquigarrow$ polynomial

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= \sum_{k=0} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The important message of Taylor's Theorem is this:
Knowing the derivatives of a function at any point a allows you to approximate the function.

Example: Approximating the Exponential Function

Approximate $f(x) = e^x$ at $a = 0$ using Taylor's Theorem in One-Dimension.
How close is the approximation at $x = 1$ after taking ten terms of the sum?

compute derivatives of e^x.

$$f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x \dots \Rightarrow f^{(n)}(x) = e^x$$

We get $f^{(n)}(a) = f^{(n)}(0) = e^0 = 1$.

Apply the formula for Taylor's Thm

$$f(x) \approx f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \dots$$

$$\approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

Taking $n \to \infty$ we get: $\boxed{e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}}$

How accurate is the approximation by the first 10 terms of the series?

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!} = \frac{9864101}{3628800} \approx 2.718281801146\dots$$

$$e^1 = e \approx \underbrace{2.7182818}_{\text{seven decimal places}}28459\dots$$

Taylor's Theorem in One-Dimension (h-version) (M&T p. 159)

Our textbook gives the following equivalent formulation of Taylor's Theorem:

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}h^k + R_k(x_0,h)$$

The remainder term $R_k(x_0,h)$ satsifies $\lim_{h\to 0} \frac{R_k(x_0,h)}{h^k} = 0$.
Intuitively, this means "$R_k(x_0,h)$ goes to zero faster than h^k."

Example: Verify That R_k Goes to Zero

Approximate $f(x) = e^x$ at $x_0 = 0$ and check that $\lim_{h\to 0} \frac{R_k(x_0,h)}{h^k} = 0$ for all k.

Big Idea: Factor out h^{k+1} and bound the left-over terms by e^h.

We have:

$$f(0+h) = e^0 + e^0 h + \frac{e^0}{2!}h^2 + \dots + \frac{e^0}{k!}h^k + R_k(x_0,h)$$

$$e^h = 1 + h + \frac{1}{2!}h^2 + \dots + \frac{1}{k!}h^k + R_k(x_0,h)$$

Use the Taylor expansion to find $R_k(x_0,h)$.

We have $R_k(x_0,h) = \frac{1}{(k+1)!}h^{k+1} + \frac{1}{(k+2)!}h^{k+2} + \dots$

= "the remaining terms of the infinite sum"

Bound each term in the brackets by the corresponding term of e^h

$$= h^{k+1}\left(\frac{1}{(k+1)!} + \frac{h}{(k+2)!} + \frac{h^2}{(k+3)!} + \dots\right)$$

$$\leq h^{k+1}\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots\right)$$

$$= h^{k+1}e^h$$

This gives $\lim_{h\to 0} \frac{R_k(0,h)}{h^k} \leq \lim_{h\to 0} \frac{h^{k+1}e^h}{h^k} = \lim_{h\to 0} he^h = 0.$

Differentibility (M&T p. 111)

Let $\mathbf{D} = \mathbf{D}f(\mathbf{x}_0)$. We say that f is differentiable at x_0 if the following limit exists:

$= Df(x_0)$

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0$$

This is like saying: $f(x) - f(x_0) \approx D(x - x_0)$

Note: The product $\mathbf{D}(\mathbf{x}-\mathbf{x}_0)$ is a matrix $\mathbf{D}$ times the column vector $\mathbf{x}-\mathbf{x}_0$.

Example: A Quick Verification

Verify that $f(x,y) = x^2 + y^2$ is differentiable at $(0,0)$. $f: \mathbb{R}^2 \to \mathbb{R}^1$

Calculate the matrix D

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 2y \end{bmatrix} \to Df(0,0) = [0, 0]$$

In one variable:

$$\underbrace{f(x) - f(x_0)}_{\text{number}} \approx \underbrace{f'(x_0)}_{\text{number}} \underbrace{(x - x_0)}_{\text{number}}$$

In multiple variables:

$$\underbrace{f(x) - f(x_0)}_{\text{vector}} = \underbrace{D}_{\text{matrix}} \underbrace{(x - x_0)}_{\text{vector}}$$

Setup the limit

$$\lim_{(x,y)\to(0,0)} \frac{\|x^2+y^2 - (0^2+0^2) - D(x-x_0)\|}{\|(x,y)-(0,0)\|}$$

$Df = [0\ 0]$

$$= \lim_{(x,y)\to(0,0)} \frac{\left\| x^2+y^2 - \cancel{\begin{bmatrix} 2x & 2y \end{bmatrix}} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \right\|}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\|}$$

$Df = [0\ 0]$

$$= \lim_{(x,y)\to(0,0)} \frac{\| x^2+y^2 - \cancel{(2x^2+2y^2)} \|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\|-x^2-y^2\|}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0\ 0)} \sqrt{x^2+y^2} = 0$$

First-Order Multivariable Taylor (M&T p. 160)

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{x}_0$. We have the following approximation:

$$\boxed{f(\mathbf{x}_0 + \mathbf{h})} = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + R_1(\mathbf{x}_0, \mathbf{h}) = \boxed{f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)]\mathbf{h}} + R_1(\mathbf{x}_0, \mathbf{h})$$

where $\lim_{\mathbf{h}\to 0} R_1(\mathbf{x}_0, \mathbf{h}) / \|\mathbf{h}\| = 0$. We call

We need to check that $R_1(x_0, h)$ goes to zero fast enough.

$$f(\mathbf{x}_0 + \mathbf{h}) \approx f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)]\mathbf{h}$$

the <u>first-order Taylor approximation</u> of f at $\mathbf{x}_0$.

The Big Idea: Take $R_1(\mathbf{x}_0, \mathbf{h}) = f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - [\mathbf{D}f(\mathbf{x}_0)]\mathbf{h}$. Taking $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$ we get:

diff'able $\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0 \overset{\Rightarrow}{\cancel{\Longleftrightarrow}} \lim_{\mathbf{h}\to\mathbf{0}} \frac{R_1(\mathbf{x}_0, \mathbf{h})}{\|\mathbf{h}\|} = 0$

Assume f is differentiable at x_0.

We set: $R_1(x_0, h) = \underbrace{f(x_0+h)}_{\text{function}} - \underbrace{f(x_0) - [Df(x_0)]h}_{\text{approximation}}$

We calculate

$$\lim_{h\to 0} \frac{R_1(x_0, h)}{\|h\|} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - [Df(x_0)]h}{\|h\|}$$

This is the derivative at x_0. It is a matrix Df.

We pick $h = x - x_0$ and get:

$$= \lim_{x - x_0 \to 0} \frac{f(x) - f(x_0) - [Df(x_0)](x - x_0)}{\|x - x_0\|}$$

by diff'able

Norms are upper bounds

$$\leq \lim_{x\to x_0} \frac{\|f(x) - f(x_0) - Df\,(x - x_0)\|}{\|x - x_0\|} = 0$$

In summary: $f \approx f(x_0) + [Df(x_0)]h$ is a good approximation.

The first-order Taylor approximation of $f(\mathbf{x}_0)$ at $\mathbf{x}_0$ is:

variable

$$f(\mathbf{x}_0 + \mathbf{h}) \approx f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)]\mathbf{h}$$

Example: Approximate a Nice Function

Give the first order Taylor approximation of $f(x,y) = xye^{x+y}$ at $(x,y) = (1,1)$.

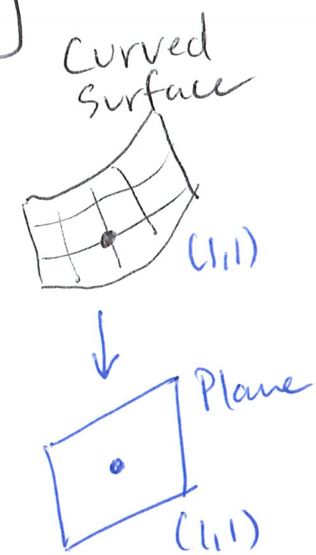
Apply the approximation formula.

Calculate $Df(x_0)$

$$Df = \left[\frac{\partial(xye^{x+y})}{\partial x}, \frac{\partial(xye^{x+y})}{\partial y}\right]$$

$$= \left[ye^{x+y} + xye^{x+y}, xe^{x+y} + xye^{x+y}\right]$$

$$Df(1,1) = \left[e^2+e^2, e^2+e^2\right] = \left[2e^2, 2e^2\right]$$



Calculate $f(x_0)$

$$f(1,1) = 1 \cdot 1 e^{1+1} = e^2$$

Putting the parts together we get:

$$f\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}h_1\\h_2\end{bmatrix}\right) \approx e^2 + \left[2e^2, 2e^2\right]\begin{bmatrix}h_1\\h_2\end{bmatrix}$$

Approximately equals →

$$\approx e^2 + 2e^2 h_1 + 2e^2 h_2$$

$$\approx e^2(1 + 2h_1 + 2h_2)$$

$$f\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}h_1\\h_2\end{bmatrix}\right) \approx e^2 + 2e^2(h_1 + h_2)$$

Quadratic Forms (M&T p. 172)

Suppose that $M = [m_{ij}]$ is an $n \times n$ matrix. The quadratic form of M is:

$$Q_M(\mathbf{h}) = \mathbf{h}^T M \mathbf{h}$$

In coordinatates:

$$Q_M(h_1, h_2, \ldots, h_n) = \begin{bmatrix} h_1 & h_2 & \ldots & h_n \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & & m_{2n} \\ \vdots & & \ddots & \\ m_{n1} & m_{n2} & & m_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} h_i h_j$$

Example: Nice Functions Arise as Quadratics

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow f_M(h_1, h_2) = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = h_1^2 - h_2^2$$

The Big Idea: Locally, every function looks like a quadratic function. "Morse Theory"

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} h_1 \\ -h_2 \end{bmatrix} = h_1^2 - h_2^2$$ ← The result is a nice quadratic function

The Hessian Function (M&T p. 172)

Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^2. The Hessian of f at $\mathbf{x}_0$ is the quadratic function:

Recall: $f_{x_i x_j} = f_{x_j x_i}$.

$$Hf(\mathbf{x}_0)(\mathbf{h}) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) h_i h_j$$

$$= \frac{1}{2}[h_1, h_2, \ldots, h_n] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}_0) & \boxed{\frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}_0)} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}_0) \\ \boxed{\frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}_0)} & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}_0) \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{x}_0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

Notice: The Hessian is analogous to the second-derivative of f. This matrix is symmetric!

Example: Hessian of a Saddle

Compute $Hf(\mathbf{x}_0)(\mathbf{h})$ for $f(x, y) = x^2 - y^2$.

$$Hf(x_0)(h) = \frac{1}{2}[h_1 \; h_2] \begin{bmatrix} \frac{\partial^2}{\partial x \partial x}(x^2 - y^2) & \frac{\partial^2}{\partial x \partial y}(x^2 - y^2) \\ \frac{\partial^2}{\partial y \partial x}(x^2 - y^2) & \frac{\partial^2}{\partial y \partial y}(x^2 - y^2) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= \frac{1}{2}[h_1, h_2] \begin{bmatrix} \frac{\partial}{\partial x}(2x) & \frac{\partial}{\partial x}(-2y) \\ \frac{\partial}{\partial y}(2x) & \frac{\partial}{\partial y}(-2y) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= \frac{1}{2}[h_1 \; h_2] \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2}(2h_1^2 - 2h_2^2) = h_1^2 - h_2^2$$

Second-Order Multivariable Taylor (M&T p. 160)

If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^3 then we have::

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \underbrace{\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0)}_{Df(x_0)h} + \underbrace{\frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)}_{Hf(h)} + R_2(\mathbf{x}_0, \mathbf{h})$$

Here, the term $R_2(\mathbf{x}_0, \mathbf{h})$ goes to zero faster than $\|h\|^2$. Formally, $\lim_{\|h\| \to 0} \frac{R_2(\mathbf{x}_0, \mathbf{h})}{\|h\|^2} = 0$.

We define the second-order Taylor approximation of f at $\mathbf{x}_0$ to be:

$$f(\mathbf{x}_0 + \mathbf{h}) \approx f(\mathbf{x}_0) + [\mathbf{D}f(\mathbf{x}_0)]\mathbf{h} + Hf(\mathbf{h})$$

Example: Second-Order Taylor of a Quadratic

Compute the second-order Taylor polynomial of $f(x, y) = x^2 + 2x + 3y + 4xy + 5y^2$ at $(x, y) = (0, 0)$. *Notice*: The final answer ought to be $f(x, y)$ again because $f(x, y)$ is a polynomial of degree two.

$$Hf = \frac{1}{2}[h_1\ h_2]\begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \quad \text{"} x^2 + 4xy + 5y^2 \text{"}$$

$$Df = [2x + 2 + 4y,\ 3 + 4x + 10y] \longrightarrow Df(0,0) = [2, 3]$$

"$2x + 3y$"

Calculation Completed After Lecture

This gives:

$$f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) \approx f(0,0) + [2\ 3]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \frac{1}{2}[h_1\ h_2]\begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= 0 + 2h_1 + 3h_2 + \frac{1}{2}[h_1\ h_2]\begin{bmatrix} 2h_1 + 4h_2 \\ 4h_1 + 10h_2 \end{bmatrix}$$

$$= 2h_1 + 3h_2 + \frac{1}{2}\left[h_1(2h_1 + 4h_2) + h_2(4h_1 + 10h_2)\right]$$

$$= 2h_1 + 3h_2 + \frac{1}{2}(2h_1^2 + 4h_1h_2 + 4h_2h_1 + 10h_2^2)$$

$$= 2h_1 + 3h_2 + h_1^2 + 4h_1h_2 + 5h_2^2$$

OMG! It looks so much like the original polynomial

$$= h_1^2 + 2h_1 + 3h_2 + 4h_1h_2 + 5h_2^2$$