Week 7: Constrained Optimization

In the last example (p80) $g(x,y) = x^2 + 2y^2 = 1$.

Restriction (M&T p. 186)

Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a function and a set $S \subset \mathbb{R}^n$. The restriction of f to S is $f|S : S \to \mathbb{R}$.

Objectives and Constraints (M&T p. 186)

Suppose that f and g are functions from $\mathbb{R}^n$ to $\mathbb{R}$.
If we want to find the local extrema of f subject to $g = c$ then we say:
f is the objective function and g is the constraint.

$g=c$ is a level set L_c

$f|L_c$ might not be constant.

Activity: Make Some Examples

Give some examples of real situations where you would want to maximixe f subject to $g = c$.
For each example, you need both an objective and a constraint.

- heighest point of $\underbrace{\text{waves}}_{f}$ in a $\underbrace{\text{tank}}_{g}$
- heighest point on a $\underbrace{\text{trail}}_{g}$ on a $\underbrace{\text{mountain}}_{f}$
- maximize $\underbrace{\text{profit}}_{f}$ from $\underbrace{\text{two sources of income}}_{g}$

☆ The Lagrange Multiplier Method (M&T p. 185)

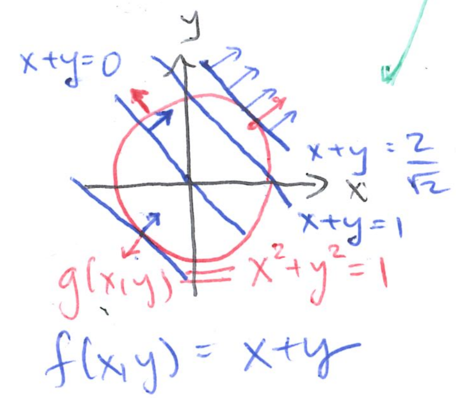
Suppose that f and g are C^1 functions from $U \subset \mathbb{R}^n$ to $\mathbb{R}$. Let $\mathbf{x}_0 \in U$ be a point satisfying $g(\mathbf{x}_0) = c$ and $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$. Furthermore, let S be the level set $L_c = \{\mathbf{x} : g(\mathbf{x}) = c\} = S$

If $f|S$ has a local extrema at $\mathbf{x}_0$ then: $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$ ← gradients are parallel

The Big Idea: At a local extrema, the objective must be tangent to the constraint. Informally, if there was not tangency, you could increase or decrease value of c while still meeting the constraint. Formally, suppose that $\mathbf{c}(t)$ is a curve in S passing through $\mathbf{x}_0$ at $t = 0$. We have $g(\mathbf{c}(t)) = c$. It follows that:

$$0 = \frac{d}{dt}\left[g(\mathbf{c}(t))\right] = \nabla g(\mathbf{x}_0) \cdot \mathbf{c}'(0)$$

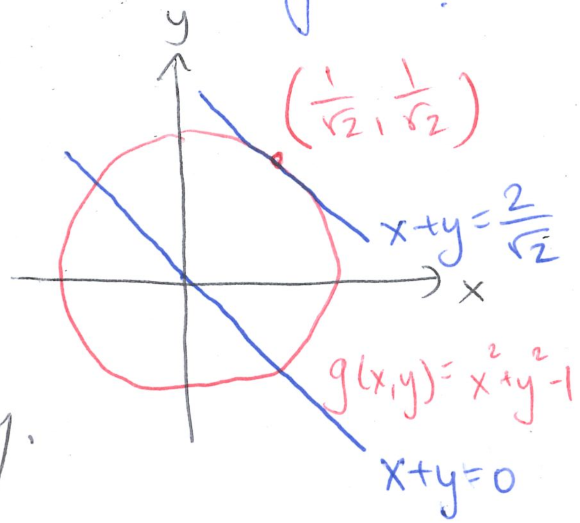
On the other hand, at an extrema, we have $0 = \frac{d}{dt}\left[f(\mathbf{c}(t))\right]$ from single-variable calculus.
Thus, ∇f is perpendicular to all curves through $\mathbf{x}_0$. It follows that $\nabla g(\mathbf{x}_0)$ and $\nabla f(\mathbf{x}_0)$ must be parallel.



Example: The Level Curves Interpretation of Lagrange

Find the maximum of $f(x, y) = x + y$ subject to $g(x, y) = x^2 + y^2 - 1 = 0$ by drawing the level sets of $f(x, y)$ and finding the largest c such that L_c is tangent to $g(x, y) = 0$.

This is the exact same setup as the diagram!



Notice:

Any value of $c > \frac{2}{\sqrt{2}} = \sqrt{2}$ will not satify $x^2+y^2-1=0$.

There will be no tangency.

The Lagrange Auxilliary Function (M&T p. 187)

Given an objective f and a constraint $g = c$, we define the following Lagrange auxilliary function:

$$\mathcal{L}(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, x_2, \ldots, x_n) - \lambda\,(g(x_1, \ldots, x_n) - c)$$

The condition $\nabla \mathcal{L} = \mathbf{0}$ is equivalent to:

$$\begin{cases} \left.\begin{aligned} \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} &= 0 \\ &\vdots \\ \frac{\partial f}{\partial x_n} - \lambda \frac{\partial g}{\partial x_n} &= 0 \end{aligned}\right\} \Leftrightarrow \nabla f = \lambda \nabla g \\ g(x_1, \ldots, x_n) - c = 0 \Leftrightarrow g = c \end{cases}$$

Warning: This condition says where an extrema might exist, but doesn't guarantee existence. Moreover, this system of equations is generally non-linear. Expect magic!

Example: Lagrange on the Sphere

Setup the Lagrange auxilliary function for the following problem:
Find the extreme values of $f(x, y, z) = x + z$ on the sphere $x^2 + y^2 + z^2 = 1$.

$$\begin{aligned} \mathcal{L} &= f - \lambda(g - c) \\ &= (x+z) - \lambda(x^2+y^2+z^2-1) \end{aligned}$$

Example: Lagrange on the Sphere

Find the extreme values of $f(x, y, z) = x + z$ on the sphere $x^2 + y^2 + z^2 = 1$.

Setup Lagrange auxillary function.

$$\mathcal{L} = (x + z) - \lambda(x^2 + y^2 + z^2 - 1)$$

Find critical points of $\mathcal{L}$.

$$\nabla \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial y}, \frac{\partial \mathcal{L}}{\partial z}, \frac{\partial \mathcal{L}}{\partial \lambda}\right) = (0, 0, 0, 0)$$

$$= (1 - 2x\lambda, 0 - 2y\lambda, 1 - 2z\lambda, -(x^2 + y^2 + z^2 - 1))$$

As a system of non-linear equations:

$$\begin{cases} 1 - 2x\lambda = 0 \\ 0 - 2y\lambda = 0 \\ 1 - 2z\lambda = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases}$$

~~If $x, y, z \neq 0$?~~

~~$\lambda = \frac{1}{2x} = \frac{1}{2z}$ and $\lambda = 0$~~

Case #1: $y \neq 0$

$$-2y\lambda = 0 \Rightarrow \lambda = 0$$

This gives $1 - 2x\lambda = 1 - 0 = 0$

This contradicts the constraint.

oh no!

Case #2: $y = 0$

We get: $x^2 + y^2 + z^2 - 1 = 0 \Rightarrow x^2 + z^2 = 1$.

We now use $x, z \neq 0$ to get:

$$0 = 1 - 2x\lambda \Leftrightarrow x = \frac{1}{2\lambda}$$

$$0 = 1 - 2z\lambda \Leftrightarrow z = \frac{1}{2\lambda}$$

It follows: $x^2 + z^2 = 1 \Leftrightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \Leftrightarrow \frac{2}{4\lambda^2} = 1$

$$\Leftrightarrow \lambda = \pm \frac{1}{\sqrt{2}}.$$

We conclude, the extrema are at $(x, y, z) = \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$

The max is $(x, 0, z) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and min is $(x, 0, z) = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

Example: Lagrange on the Disk

Find the extreme values of $f(x,y) = x^2 + xy + y^2$ on the disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$.

We need to look for extrema in the interior of D ← unconstrained
and on the boundary ∂D ← Lagrange

Find critical points in the interior of D:

$$\nabla f = (2x+y, x+2y) = (0,0) \iff \begin{cases} 2x+y=0 \\ x+2y=0 \end{cases}$$
$$\iff (x,y) = (0,0).$$

Classify using second deriv test.

$$Hf = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \implies \begin{matrix} \det[2] > 0 \\ \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0 \end{matrix} \implies (0,0) \text{ is a min.}$$

Find extrema on the boundary:

$$\mathcal{L} = x^2 + xy + y^2 - \lambda(x^2+y^2-1)$$

$$\nabla \mathcal{L} = 0 \iff \begin{cases} 2x + y - 2x\lambda = 0 & (1) \\ x + 2y - 2y\lambda = 0 & (2) \\ x^2 + y^2 - 1 = 0 & (3) \end{cases}$$

We add equations (1) and (2) to get:

$$3x + 3y - 2(x+y)\lambda = 0$$
$$\iff 3(x+y) = 2(x+y)\lambda$$

Case #1: $x+y=0$

(1) $2x+y-2x\lambda=0$
$\iff x - 2x\lambda = 0$
$\iff x(1-2\lambda) = 0$

In this case $x=-y$.

Case #2: $x+y \neq 0$

$3(x+y) = 2(x+y)\lambda \iff \lambda = \frac{3}{2}$

(1) $2x+y-3x=0 \iff -x+y=0$

(2) $x+2y-3y=0 \iff x-y=0$

Thus: $x=y$

continued on back of p. 85

Case #1: $x+y=0$

$\Rightarrow x=-y$

Case #2: $x+y\neq 0$

$\Rightarrow y=x$

(3) $x^2+y^2-1=0$ then gives:

$x^2+(-x)^2-1=0$

$2x^2=1$

$x=\pm\frac{1}{\sqrt{2}}$

we get:

$(x,y)=\pm\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$

$$\underbrace{\begin{aligned} f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) &= \frac{1}{2}-\frac{1}{2}+\frac{1}{2} \\ &= \frac{1}{2} \\ f\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) &= \frac{1}{2}-\frac{1}{2}+\frac{1}{2} \\ &= \frac{1}{2} \end{aligned}}_{\text{Minima}}$$

$x^2+x^2-1=0$

$2x^2=1$

$x=\pm\frac{1}{\sqrt{2}}$

we get:

$(x,y)=\pm\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$$\underbrace{\begin{aligned} f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) &= \frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2} \\ f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) &= \frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2} \end{aligned}}_{\text{Maxima}}$$

Example: A Maximum without A Minimum

Find the maximum value of $f(x,y) = e^{xy}$ subject to the constraint $x^3 + y^3 = 16$. Show that it has no minimum value.

Apply Lagrange multipliers

$$\mathcal{L} = e^{xy} - \lambda(x^3+y^3-16)$$

Find critical points of $\mathcal{L}$.

$$\nabla\mathcal{L} = \Big(ye^{xy} - \lambda(3x^2),\ xe^{xy} - \lambda(3y^2),\ \underbrace{-(x^3+y^3-16)}_{\frac{\partial\mathcal{L}}{\partial\lambda}}\Big) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} ① \ ye^{xy} - 3\lambda x^2 = 0 \\ ② \ xe^{xy} - 3\lambda y^2 = 0 \\ ③ \ x^3+y^3-16 = 0 \end{cases}$$

To get more symmetry we multiply ① by x and ② by y

$$xye^{xy} - 3\lambda x^3 = 0 \Leftrightarrow xye^{xy} = 3\lambda x^3$$
$$xye^{xy} - 3\lambda y^3 = 0 \qquad xye^{xy} = 3\lambda y^3$$

Thus, $3\lambda x^3 = 3\lambda y^3$

Multiply ③ by 3λ

$$3\lambda x^3 + 3\lambda y^3 - 16\cdot 3\lambda = 0$$
$$\Leftrightarrow 6\lambda x^3 = 16\cdot 3\lambda$$

Continued on back of p. 87.

Case #1: $\lambda \neq 0$

$$x^3 = \frac{16\cdot 3}{6} = 8 \Leftrightarrow x = 2.$$

Case #2: $\lambda = 0$

① $\Rightarrow ye^{xy} = 0$

② $\Rightarrow xe^{xy} = 0$

Thus, $(x,y) = (2,2)$

Thus $(x,y) = (0,0)$

This contradicts ③.

We have a Lagrange critical point $(x,y) = (2,2)$.
Our function is $f(x,y) = e^{xy}$
At $(2,2)$ we get: $f(2,2) = e^4$

We now argue $f(x,y) = e^{xy}$ has no min on $x^3+y^3 = 16$
The curve $x^3+y^3 = 16$ is not bounded.
Consider the point $x = t$ on the curve.

$$t^3 + y^3 = 16 \Rightarrow y = \sqrt[3]{16 - t^3}$$

When t is large, $16 - t^3$ is negative.
Thus, $\sqrt[3]{16-t^3}$ is also negative.
We now argue that on $c(t) = \left(t, \sqrt[3]{16-t^3}\right)$
the function $f(x,y)$ is arbitrarily close to zero.

$$f(c(t)) = e^{t\sqrt[3]{16-t^3}} \leq e^{-t} \longrightarrow 0.$$

This inequality holds to t large and positive.

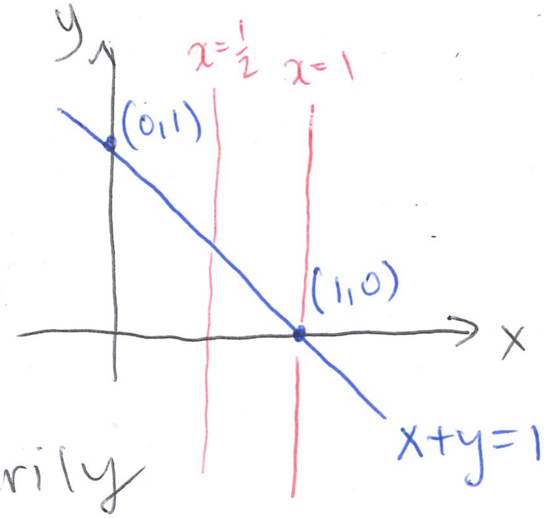
Example: When Lagrange Goes Wrong

Find the extreme values of $f(x, y) = x$ subject to the constraint $x + y = 1$.

The level set $L_c = \{(x,y) : x = c\}$ intersects the constraint

$$x + y = 1$$

for all values of c.



We can make $f(x,y)$ arbitrarily large AND arbitrarily small while still satisfying the constraint.

Setup Lagrange

$$\mathcal{L} = x - \lambda(x+y-1)$$

$$\nabla \mathcal{L} = 0 \iff \begin{cases} 1 - \lambda = 0 \\ -\lambda = 0 \\ x+y-1 = 0 \end{cases}$$

This system is inconsistent! It has NO solutions.

Activity: Discuss (5 min)

Does the function $f(x, y) = xy\exp(-x^2 - y^2)$ have a minimum or maximum subject to $2x - y = 0$?

$\Leftrightarrow y = 2x$

On the constraint:

$$f(x,y) = f(x,2x) = x(2x)e^{-x^2-(2x)^2}$$

$$= \underbrace{2x^2}_{\geq 0}\underbrace{e^{-5x^2}}_{\geq 0} \geq 0 \text{ because } x^2 \geq 0 \text{ and } e^t \geq 0$$

$$f(0,0) = 2\cdot 0^2 e^{-5\cdot 0^2} = 0 \leftarrow \text{Global minimum.}$$

This has a global max as well.
You can find it using single variable calc.

In this example:
We have an unbounded constrain $y = 2x$.
Also, we have global max/min.
In the case of unbounded constraints we need to be careful.

Lagrange with Multiple Constraints (M&T p. 191)

Given multiple constraints $g_1 = c_1, g_2 = c_2, \ldots, g_k = c_k$ such that $\{\nabla g_1(\mathbf{x}_0), \ldots \nabla g_k(\mathbf{x}_0)\}$ are linearly independent. Let S be the set where all the constraints are satisfied and $\mathbf{x}_0 \in S$. The method of Lagrange multipliers generalizes to multiple constraints as follows.

If $f|S$ has a local extrema at $\mathbf{x}_0$ then there are constants such that:

Linear combo.

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \lambda_2 \nabla g_2(\mathbf{x}_0) + \cdots + \lambda_k \nabla g_k(\mathbf{x}_0)$$

Example: A Slice of a Cylinder

Use the method of Lagrange multipliers to find the extreme values of $f(x, y, z) = x + 2y + 3z$ on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x - y + z = 1$.

Setup Lagrange

$$\mathcal{L} = x + 2y + 3z - \lambda_1(x^2 + y^2 - 1) - \lambda_2(x - y + z - 1)$$

$$\nabla \mathcal{L} = 0 \Leftrightarrow \begin{cases} 1 - 2x\lambda_1 - \lambda_2 = 0 & (1) \\ 2 - 2y\lambda_1 + \lambda_2 = 0 & (2) \\ 3 - 0 - \lambda_2 = 0 & (3) \Leftrightarrow \lambda_2 = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} -2 - 2x\lambda_1 = 0 \\ 5 - 2y\lambda_1 = 0 \end{cases}$$

From here: Argue based on cases whether $\lambda_1 = 0$ or $\lambda_1 \neq 0$.

To be continued.

We get $(x, y, z) = \left(\frac{\mp 2}{\sqrt{29}}, \frac{\pm 5}{\sqrt{29}}, 1 \pm \frac{7}{\sqrt{29}}\right)$.