Lagrange with Multiple Constraints (M&T p. 191)

Given multiple constraints $g_1 = c_1, g_2 = c_2, \ldots, g_k = c_k$ such that $\{\nabla g_1(\mathbf{x}_0), \ldots \nabla g_k(\mathbf{x}_0)\}$ are linearly independent. Let S be the set where all the constraints are satisfied and $\mathbf{x}_0 \in S$. The method of Lagrange multipliers generalizes to multiple constraints as follows.

If $f|S$ has a local extrema at $\mathbf{x}_0$ then there are constants such that:

Linear combo.

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \lambda_2 \nabla g_2(\mathbf{x}_0) + \cdots + \lambda_k \nabla g_k(\mathbf{x}_0)$$

Example: A Slice of a Cylinder

Use the method of Lagrange multipliers to find the extreme values of $f(x, y, z) = x + 2y + 3z$ on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x - y + z = 1$.

Setup Lagrange

$$\mathcal{L} = x + 2y + 3z - \lambda_1(x^2+y^2-1) - \lambda_2(x-y+z-1)$$

$$\nabla \mathcal{L} = 0 \Leftrightarrow \begin{cases} 1 - 2x\lambda_1 - \lambda_2 = 0 & (1) \\ 2 - 2y\lambda_1 + \lambda_2 = 0 & (2) \\ 3 - 0 - \lambda_2 = 0 & (3) \Leftrightarrow \lambda_2 = 3 \end{cases}$$

$x^2+y^2-1=0$ (4)

$x-y+z-1=0$ (5)

$$\Leftrightarrow \begin{cases} -2-2x\lambda_1 = 0 & (1) \\ 5-2y\lambda_1 = 0 & (2) \end{cases}$$

From here: Argue based on cases whether $\lambda_1 = 0$ or $\lambda_1 \neq 0$.

Case 1: $\lambda_1 = 0$

(1) $\Rightarrow -2 = 0$

contradiction!

Case 2: $\lambda_1 \neq 0$

(1) $\Rightarrow x = \frac{-1}{\lambda_1}$

(2) $\Rightarrow y = \frac{5}{2\lambda_1}$

Putting these in (4) we get:

$$\left(\frac{-1}{\lambda_1}\right)^2 + \left(\frac{5}{2\lambda_1}\right)^2 = 1$$

$$\Rightarrow 4 + 25 = 4\lambda_1^2$$

$$\Rightarrow \lambda_1^2 = \frac{29}{4}$$

Lagrange for linear functions is nice.

We get $(x, y, z) = \left(\frac{\mp 2}{\sqrt{29}}, \frac{\pm 5}{\sqrt{29}}, 1 \pm \frac{7}{\sqrt{29}}\right)$.

We have:

$$\lambda_1^2 = \frac{29}{4} \Rightarrow \lambda_1 = \frac{\pm\sqrt{29}}{2}$$

From $x = \frac{-1}{\lambda_1}$ we get: $x = \mp \frac{2}{\sqrt{29}}$

$y = \frac{5}{2\lambda_1}$ we get $y = \frac{5}{2\,\frac{\pm\sqrt{29}}{2}} = \pm\frac{5}{\sqrt{29}}$

From $x - y + z = 1$ we get: $z = 1 - x + y$ and

$$z = 1 \pm \frac{2}{\sqrt{29}} \pm \frac{5}{\sqrt{29}}$$

We input these points in to $f(x,y,z) = x + 2y + 3z$

We choose to only consider positive inputs when finding our maximum.

$$f\left(\frac{-2}{\sqrt{29}}, \frac{+5}{\sqrt{29}}, 1 + \frac{2}{\sqrt{29}} - \frac{5}{\sqrt{29}}\right) \leftarrow \text{Maximum}$$

$$f\left(\frac{+2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, 1 - \frac{2}{\sqrt{29}} - \frac{5}{\sqrt{29}}\right) \leftarrow \text{Minimum}$$

Agg! Tricky signs.
See bottom of p. 90.

Example: A Slice of a Cone

Use the method of Lagrange multipliers to find the extreme values of $f(x, y, z) = z$ on the intersection of the cone $x^2 + y^2 = z^2$ and the plane $x + y + z = 24$.

Setup Lagrange

$$\mathcal{L} = z - \lambda_1(x^2+y^2-z^2) - \lambda_2(x+y+z-24)$$

$$\nabla\mathcal{L} = 0 \iff \begin{cases} -2x\lambda_1 - \lambda_2 = 0 & (1) \\ -2y\lambda_1 - \lambda_2 = 0 & (2) \\ 1+2z\lambda_1 - \lambda_2 = 0 & (3) \\ x^2+y^2-z^2 = 0 & (4) \\ x+y+z-24 = 0 & (5) \end{cases}$$

Case #1: $\lambda_1 = 0$

(1) $\Rightarrow -\lambda_2 = 0$

(3) $\Rightarrow 1-\lambda_2 = 0$

Contradiction!

Case #2: $\lambda_1 \neq 0$

(1) $\Rightarrow x = \dfrac{\lambda_2}{-2\lambda_1}$ (2) $\Rightarrow y = \dfrac{\lambda_2}{-2\lambda_1}$

Thus, $x = y$.

From (4) we get: $x^2+y^2-z^2=0 \Rightarrow 2x^2-z^2=0$

$\Rightarrow z = \pm\sqrt{2}\,x = \pm\sqrt{2}\,y$

From (5) we get: $x+y+z-24=0 \Rightarrow 2x \pm \sqrt{2}x - 24 = 0$

we get: $(2\pm\sqrt{2})x = 24 \iff x = \dfrac{24}{2\pm\sqrt{2}}$.

Maximum

It follows from (5) that $z = 24 - x - y = 24 - \dfrac{24}{2+\sqrt{2}} - \dfrac{24}{2+\sqrt{2}}$

OR $= 24 - \dfrac{24}{2-\sqrt{2}} - \dfrac{24}{2-\sqrt{2}}$

Minimum

Week 8: Double Integrals ←

Cavalier's Principle or The Slice Method (M&T p. 266)

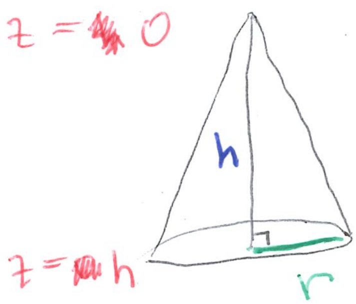
Suppose that R is a region in $\mathbb{R}^3$ contained between $z = a$ and $z = b$.
Let $A(t) = \text{Area}(R \cap \{z = t\})$ be the area of the slice of R at height t.

$$\text{Vol}(R) = \int_a^b A(t)dt$$

"Volume is integral of area"

Example: The Volume of a Cone

Find the volume of the right circular cone with base radius r and height h.



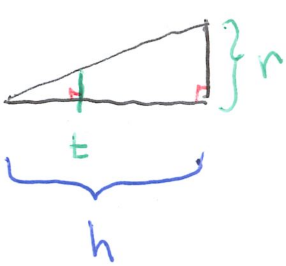
Calculate area of slices.

$$A(t) = \text{Area}(R \cap \{z = t\})$$

$$= \pi (r(t))^2 = \pi \left(\frac{r}{h} t\right)^2$$

Find radius r(t)

By similar triangles



$$\frac{r(t)}{t} = \frac{r}{h} \Rightarrow r(t) = \frac{r}{h} t$$

Calculate volume

$$V = \int_0^h A(t)dt = \int_0^h \pi \left(\frac{r}{h}\right)^2 t^2 dt$$

$$= \pi \left(\frac{r}{h}\right)^2 \int_0^h t^2 dt = \pi \left(\frac{r}{h}\right)^2 \frac{1}{3} h^3$$

$$= \frac{\pi r^2 h}{3}.$$

This is handy for things with rotational symmetry.

Double Integral (M&T p. 264)

Let $R \subset \mathbb{R}^2$ be a region and $f : R \to \mathbb{R}$ be a non-negative function. The volume above a region R and below f is given by the double integral.

$$\iint_R f(x,y)\ dA = \iint_R f(x,y)\ dxdy$$

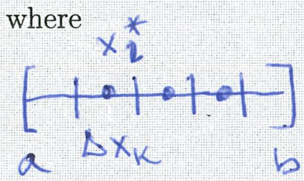
$dA = dxdy$
$= dydx$

The term dA is the area element of Cartesian coordinates.
We will say more about such terms in Weeks 11 & 12 (in polar coordinates) and MAT B42 (as differential forms).

Theorem: The Riemann Integral in One Dimension

The regular partition of $[a, b]$ in to N parts is the set of points $\{x_i\}_{i=0}^{N}$ where

$$a = x_0 < x_1 < \cdots < x_N = b$$

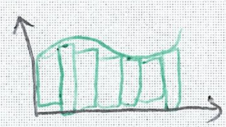


and $\Delta x_k = x_{k+1} - x_k = \dfrac{b-a}{N} = \dfrac{\text{"length"}}{N}$.

A set of points $\{x_i^*\}_{i=0}^{N}$ where $x_i^* \in [x_i, x_{i+1}]$ is called a set of sample points.

Given a function $f : [a, b] \to \mathbb{R}$ we define its Riemann integral with sample points $\{x_i^*\}_{i=0}^{N}$ to be:

$$\int_a^b f(x)dx = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i^*)\, \Delta x_i$$



It is an important theorem that for many functions this integral is independent of the choice of x_i^*.

Activity: Think-Pair-Share (3 minutes)

It is always subtle to the sample points x_i^* correctly. Suppose that $\{x_i\}_{i=0}^{N}$ is the regular partition of $[a, b]$. Give a formula (in terms of a, b, i, N) for x_i^* when using the following sample points.

- Left endpoints $x_i^* = x_i = \left(\frac{b-a}{N}\right) i + a$
- Midpoints $x_i^* = \frac{1}{2}(x_i + x_{i+1})$
- Right endpoints $x_i^* = x_{i+1} = \left(\frac{b-a}{N}\right)(i+1) + a$

Example: A Riemann Sum

Evaluate $\int_0^1 x\,dx$ using the Riemann sum definition of the definite integral with left-hand endpoints.

Write the regular partition of [0,1]

$$\Delta x_i = \frac{1-0}{N} = \frac{1}{N} \Rightarrow x_i = i\Delta x_i = i\frac{1}{N} = \frac{i}{N}$$

Write the Riemann sum

$$\sum_{i=0}^{N-1} f(x_i^*)\Delta x_i = \sum_{i=0}^{N-1} \frac{i}{N}\cdot\frac{1}{N}$$

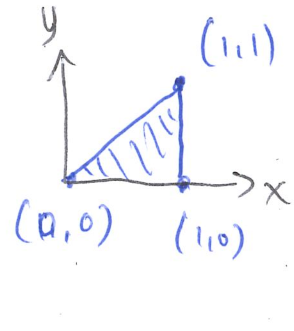
$$= \frac{1}{N^2}\sum_{i=0}^{N-1} i = \frac{1}{N^2}\frac{(N-1)(N-1+1)}{2}$$

$$= \frac{N^2-N}{2N^2}$$

Take the limit as $N \to \infty$

$$\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \lim_{N\to\infty}\frac{N^2-N}{2N^2}$$

$$= \lim_{N\to\infty}\frac{1-\frac{1}{N}}{2} = \frac{1}{2}$$



Activity: Generalize

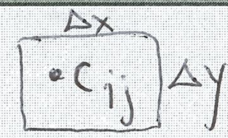
The one-dimensional Riemann sum is defined on $[a, b]$. We want to define a two-dimensional Riemann sum on $[a, b] \times [c, d]$.

- What parts of the definition need to generalize?
- What are the two-dimensional analogues of those parts?

regular partition
"litte squares"
Sample points c_{ij}

The Riemann Integral in Two Dimensions (M&T p. 272)

$$\iint_R f(x, y)\, dA = \lim_{N\to\infty} \sum_{i,j=0}^{N-1} f(\mathbf{c}_{ij}) \Delta x \Delta y$$



The Riemann Integral in Two Dimensions (M&T p. 272)

We say that f is integrable if the limit above exists and is independent of the choice of $\mathbf{c}_{ij}$.

The regular partition of $[a,b]\times[c,d]$ subdivides it in to N^2 rectangles R_{ij}.
The height will be $\frac{d-c}{N}$ and width $\frac{b-a}{N}$.
The sample point c_{ij} is in rectangle R_{ij}.

$\Delta y = \frac{d-c}{N}$

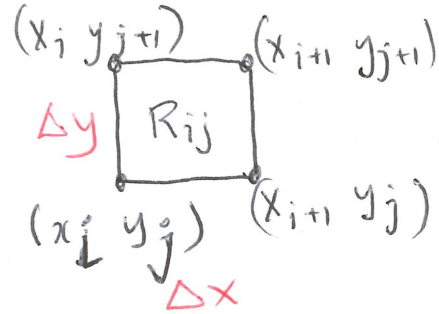
$\Delta x = \frac{b-a}{N}$

$$\sum_{i,j=0}^{N-1} f(c_{ij}) \Delta x \Delta y = \sum_{i=0}^{N-1} \left[\sum_{j=0}^{N-1} f(c_{ij}) \Delta x \Delta y \right]$$

Example: A Riemann Double-Integral

Integrate $\iint_R 2x + 3y \; dxdy$ where $R = [0,1] \times [0,1]$ using a two-dimensional Riemann sum with lower-left-corner sample points.

Pick the sample points.



$$\Delta x = \frac{1-0}{N} \qquad \Delta y = \frac{1-0}{N}$$

we get:

$$c_{ij} = (i\Delta x, j\Delta y) = \left(\frac{i}{N}, \frac{j}{N}\right)$$

Setup the Riemann Sum

$$\sum_{i,j=0}^{N-1} f(c_{ij})\Delta x \Delta y = \sum_{i,j=0}^{N-1} \left[2\left(\frac{i}{N}\right) + 3\left(\frac{j}{N}\right)\right]\Delta x \Delta y$$

$$= \sum_{i=0}^{N-1}\left[\sum_{j=0}^{N-1} 2\left(\frac{i}{N}\right) + 3\left(\frac{j}{N}\right)\right] \Delta x \Delta y$$

$$= \sum_{i=0}^{N-1}\left(\left[\sum_{j=0}^{N-1} 3\left(\frac{j}{N}\right)\right] + N\frac{2i}{N}\right)\Delta x \Delta y$$

$$= \sum_{i=0}^{N-1}\left(\left[\frac{3}{N}\sum_{j=0}^{N-1} j\right] + 2i\right)\Delta x \Delta y$$

$$= \left(3\sum_{j=0}^{N-1} j + 2\sum_{i=0}^{N-1} i\right)\Delta x \Delta y \xrightarrow{\lim N \to \infty} \frac{3}{2} + \frac{2}{2} = \frac{5}{2}$$

Example: An Integral from Data

Suppose that you know the following data about a function.

$f(x,y)$	$x=0$	$x=1$	$x=2$
$y=0$	1	2	3
$y=1$	-2	4	5
$y=2$	-3	5	6

The columns correspond to values of x and the rows correspond to values of y. The table shows the value of $f(x,y)$ at the point (x,y). For example, $f(0,2)=-3$ and $f(1,1)=4$. Use this data to estimate the following integral.

$$\int_0^2 \int_0^2 f(x,y)\,dxdy$$

Take the regular partition for $N=3$

we get $c_{ij} = (i,j)$ for $0 \leq i, j \leq 2$.

Apply the Riemann sum definition.

$$\int_0^2 \int_0^2 f(x,y)\,dxdy \approx \sum_{i=0}^{2} f(c_{ij})\Delta x \Delta y$$

$$= f(c_{00})\cdot 1 \cdot 1 + f(c_{10})\cdot 1\cdot 1 + f(c_{20})\cdot 1 \cdot 1$$
$$+ f(c_{01}) + f(c_{11}) + f(c_{21})$$
$$+ f(c_{02}) + f(c_{12}) + f(c_{22})$$

$$= (1+2+3) + (-2+4+5) + (-3+5+6)$$
$$= 6+7+8 = 21$$

Bounded Functions (M&T p. 271)

We say that a function $f(x, y)$ is <u>bounded</u> if there is a constant $M > 0$ such that $-M \leq f(x, y) \leq M$ for all points in the domain of f. $\iff |f(x,y)| \leq M$

Example: Proving a Bound

Suppose that R is a rectangle, $f(x, y)$ is integrable and bounded by $|f(x, y)| \leq M$ on R.

$$-M\,\text{Area}(R) \leq \iint_R f(x, y)\,dA \leq M\,\text{Area}(R)$$

Apply Riemann sum definition

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \sum_{i,j=0}^{N-1} f(c_{ij})\Delta x \Delta y$$

For each N we have:

$$\sum_{i,j=0}^{N-1} -M\Delta x\Delta y \leq \sum_{i,j=0}^{N-1} f(c_{ij})\Delta x\Delta y \leq \sum_{i,j=0}^{N-1} M\Delta x\Delta y$$

$$= M\underbrace{\left[\sum_{i,j=0}^{N-1}\Delta x\Delta y\right]}_{\text{Area of } R_{ij}}$$

$$-M\,\text{Area}(R) \leq \sum_{i,j=0}^{N-1} f(c_{ij})\Delta x\Delta y \leq M\,\text{Area}(R) \quad \# R = \bigcup R_{ij}$$

In the limit, we get:

$$-M\,\text{Area}(R) \leq \iint_R f(x,y)\,dA \leq M\,\text{Area}(R)$$

Properties of Integrals (M&T p. 275)

Suppose that f and g are integrable functions on R.

Linearity

$$\iint_R af(x,y) + bg(x,y)\ dA = a\iint_R f(x,y)\ dA + b\iint_R g(x,y)\ dA$$

Linear Algebra!

Monotonicity If $f(x,y) \geq g(x,y)$ then:

$$\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA$$

At sample points: $f(c_{ij}) \geq g(c_{ij})$

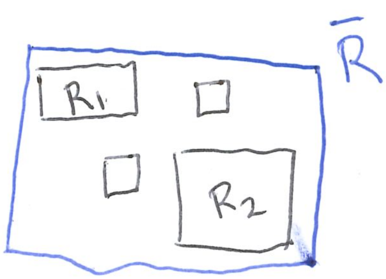
Additivity Suppose that R is the disjoint union of finitely many rectangles $R = \bigcup_{i=1}^n R_i$.

$$\boxed{\int_a^c = \int_a^b + \int_b^c}$$ 1-dim

$$\iint_R f(x,y)\ dA = \sum_{i=1}^n \iint_{R_i} f(x,y)\ dA$$

Activity: Think-Pair-Share (3 minutes)

Pick one of the properties above and convince your neighbour that it is true.



A finite collection of disjoint rectangles will be bounded.

All the rectangles fit in a big rectangle $\bar{R}$.

Consider extending f to $\bar{R}$. We let:

$$\bar{f} = \begin{cases} f(x) & \text{if } x \in R \\ 0 & \text{if } x \notin R \end{cases}$$

We get:

$$\iint_{\bar{R}} \bar{f}\, dA = \iint_R f\, dA$$

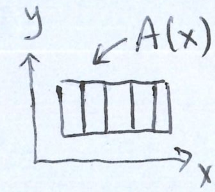
↑ Apply Riemann sums here.

group terms of the Riemann sum over $\bar{R}$ based on which rectangle c_{ij} is in. This gives:

$$\sum_i \iint_{R_i} f(x,y)\, dA$$

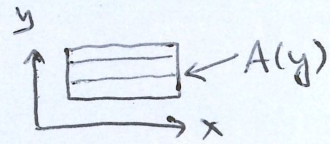
Iterated Integrals (M&T p. 267)

Suppose that $R = [a, b] \times [c, d]$ is a rectangle and f is integrable on R.
We apply Cavalier's principle, and get the following two results:
If we cut the volume perpendicular to the x-axis then



$$\iint_R f(x, y)\, dA = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

If we cut the volume perpendicular to the y-axis then



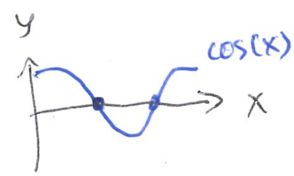
$$\iint_R f(x, y)\, dA = \int_c^d \left[\int_a^b f(x, y) dx \right] dy.$$

Example: An Integral by Slices

Calculate the following integral.

$$\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\, dxdy$$

Integrate with respect to x.



$$\int_0^{\pi/2} \Big[-\cos(x+y) \Big]_{x=0}^{x=\pi/2} dy$$

$$= \int_0^{\pi/2} -\cos\left(\tfrac{\pi}{2}+y\right) + \cos(0+y)\, dy \quad \# \cos\left(\tfrac{\pi}{2}+\theta\right) = -\sin(\theta)$$

$$= \int_0^{\pi/2} \sin(y) + \cos(y)\, dy$$

Integrate with respect to y.

$$= \Big[-\cos(y) + \sin(y) \Big]_{y=0}^{y=\frac{\pi}{2}}$$ Keep track of variable.

$$= \left[-\cos\left(\tfrac{\pi}{2}\right) + \sin\left(\tfrac{\pi}{2}\right) \right] - \left[-\cos(0) + \sin(0) \right]$$

$$= [0 + 1] - [-1 + 0] = 2$$

☆ Fubini's Theorem (M&T p. 279)

Let f be an integrable function with rectangular domain $R = [a,b] \times [c,d]$.

$$\int_a^b \int_c^d f(x,y)\; dydx = \int_c^d \int_a^b f(x,y)\; dxdy = \iint_R f(x,y)\; dA$$

This theorem holds for general domains R we will often apply it to non-rectangles.
Notice: This is the integration version ("$dxdy = dydx$") of the equality of mixed partials.

Example: Check Fubini's Theorem

Verify the conclusion of Fubini's theorem by calculating both orders of integration $dxdy$ and $dydx$ of

$$\iint_R ye^x\, dA$$

on the rectangle $R = [0,1] \times [2,3]$.

\# Calculate dxdy order

$$\int_2^3 \int_0^1 ye^x\, dx\, dy$$

$$= \int_2^3 \left[ye^x\right]_{x=0}^{x=1} dy$$

$$= \int_2^3 ye^1 - ye^0\, dy$$

$$= (e^1 - 1)\int_2^3 y\, dy$$

$$= (e^1 - 1)\left[\tfrac{1}{2}y^2\right]_{y=2}^{y=3}$$

$$= (e-1)\cdot \tfrac{1}{2} \cdot (3^2 - 2^2)$$

\#. Calculate dydx order

$$\int_0^1 \int_2^3 ye^x\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2e^x\right]_{y=2}^{y=3} dx$$

$$= \int_0^1 \tfrac{1}{2}\cdot 3^2 e^x - \tfrac{1}{2}\cdot 2^2 e^x\, dx$$

$$= \left(\tfrac{1}{2}\right)(3^2 - 2^2)\int_0^1 e^x\, dx$$

$$= \left(\tfrac{1}{2}\right)(3^2 - 2^2)\left[e^x\right]_{x=0}^{x=1}$$

$$= \left(\tfrac{1}{2}\right)(3^2 - 2^2)(e-1)$$